

DIFFERENTIAL GEOMETRY OF FINSLER SPACE WITH SPECIAL METRICS

A Thesis submitted to Gujarat Technological University

for the Award of

Doctor of Philosophy

in

Science-Mathematics

by

Patel Dhruvisha Pravinbhai

Enrollment No. 229999906001

under the supervision of

Dr. Brijesh Kumar A. Tripathi

Assistant Professor, Mathematics,
L. D. College of Engineering, Ahmedabad-380015, Gujarat.



GUJARAT TECHNOLOGICAL UNIVERSITY

AHMEDABAD

October-2026

DIFFERENTIAL GEOMETRY OF FINSLER SPACE WITH SPECIAL METRICS

A Thesis submitted to Gujarat Technological University

for the Award of

Doctor of Philosophy

in

Science-Mathematics

by

Patel Dhruvisha Pravinbhai

Enrollment No. 229999906001

under the supervision of

Dr. Brijesh Kumar A. Tripathi

Assistant Professor, Mathematics,
L. D. College of Engineering, Ahmedabad-380015, Gujarat.



GUJARAT TECHNOLOGICAL UNIVERSITY

AHMEDABAD

October-2026

©[Patel Dhruvisha Pravinbhai]

DECLARATION

I declare that the thesis entitled **Differential Geometry of Finsler Space with Special Metrics** submitted by me for the degree of Doctor of Philosophy is the record of research work carried out by me during the period from **January-2023 to June-2026** under the supervision of **Dr. Brijesh Kumar A. Tripathi** and this has not formed the basis for the award of any degree, diploma, associateship, fellowship, titles in this or any other University or other institution of higher learning.

I further declare that the material obtained from other sources has been duly acknowledged in the thesis. I shall be solely responsible for any plagiarism or other irregularities, if noticed in the thesis.

Signature of the Research Scholar: *D. P. Patel*

Date: *09/10/2026*

Name of Research Scholar: **Patel Dhruvisha Pravinbhai**

Place: Ahmedabad.

CERTIFICATE

I certify that the work incorporated in the thesis **Differential Geometry of Finsler Space with Special Metrics** submitted by **Ms. Patel Dhruvisha Pravinbhai** was carried out by the candidate under my supervision/guidance. To the best of my knowledge: (i) the candidate has not submitted the same research work to any other institution for any degree/diploma, Associateship, Fellowship or other similar titles (ii) the thesis submitted is a record of original research work done by the Research Scholar during the period of study under my supervision, and (iii) the thesis represents independent research work on the part of the Research Scholar.

Signature of the Research Supervisor:


9/10/2026

Date: 9/10/2026

Name of Supervisor: **Dr. Brijesh Kumar A. Tripathi**

Place: L. D. College of Engineering, Ahmedabad.

Course-work Completion Certificate

This is to certify that **Ms. Patel Dhruvisha Pravinbhai** enrollment no. **229999906001** is enrolled for PhD program in the faculty of **Science-Mathematics** of Gujarat Technological University, Ahmedabad.

(Please tick the relevant option(s))

- ☐ He/She has been exempted from the course-work (successfully completed during M.Phil Course)
- ☐ He/She has been exempted from Research Methodology Course only (successfully completed during M.Phil Course)
- ☒ He/She has successfully completed the PhD course work for the partial requirement for the award of PhD Degree. His/ Her performance in the course work is as follows-

Grade Obtained in Research Methodology [PHD2201]	Grade obtained in Research Publication and Ethics [PHD2202]	Grade Obtained in Self-Study Course/ Contact Program [PHD2203]
AB	AB	AB

Supervisor's Sign


9/10/2026

(Dr. Brijesh Kumar A. Tripathi)

Dhruvisha Patel

dhruvisha thesis Final.pdf

 Ph.D. Thesis_229999906001_Dhruvisha

 LDCE

 Gujarat Technological University

Document Details

Submission ID

trn:oid::1:3589602392

217 Pages

Submission Date

Jun 8, 2026, 1:58 PM GMT+5:30

49,574 Words

Download Date

Jun 8, 2026, 2:18 PM GMT+5:30

215,766 Characters

File Name

dhruvisha_thesis_Final.pdf

File Size

1.2 MB

D.P. Patel

BA

8% Overall Similarity

The combined total of all matches, including overlapping sources, for each database.

Filtered from the Report

- Bibliography
- Quoted Text
- Cited Text
- Small Matches (less than 10 words)

Exclusions

- 123 Excluded Matches

Match Groups

- **176 Not Cited or Quoted 8%**
Matches with neither in-text citation nor quotation marks
- **0 Missing Quotations 0%**
Matches that are still very similar to source material
- ≡ **0 Missing Citation 0%**
Matches that have quotation marks, but no in-text citation
- ⚙ **0 Cited and Quoted 0%**
Matches with in-text citation present, but no quotation marks

Top Sources

- 6% 🌐 Internet sources
- 5% 📖 Publications
- 1% 👤 Submitted works (Student Papers)

D.P. Patel

DB

Originality Report Certificate

It is certified that PhD Thesis titled **Differential Geometry of Finsler Space with Special Metrics** by **Ms. Patel Dhruvisha Pravinbhai** has been examined by us. We undertake the following:

- a. Thesis has significant new work / knowledge as compared already published or are under consideration to be published elsewhere. No sentence, equation, diagram, table, paragraph or section has been copied verbatim from previous work unless it is placed under quotation marks and duly referenced.
- b. The work presented is original and own work of the author (i.e. there is no plagiarism). No ideas, processes, results or words of others have been presented as Author own work.
- c. There is no fabrication of data or results which have been compiled/ analyzed.
- d. There is no falsification by manipulating research materials, equipment or processes, or changing or omitting data or results such that the research is not accurately represented in the research record.
- e. The thesis has been checked using **Turnitin software** (copy of originality report attached) and found within limits as per GTU Plagiarism Policy and instructions issued from time to time (i.e. permitted similarity index $\leq 10\%$).

Signature of the Research Scholar: **D. P. Patel**

Date: **09/10/2026**

Name of the Research Scholar: **Patel Dhruvisha Pravinbhai**

Signature of the Research Supervisor:



Date: **9/10/2026**

Name of the Research Supervisor: **Dr. Brijesh Kumar A. Tripathi**

Place: L. D. College of Engineering, Ahmedabad

Ph.D. Thesis Non-Exclusive License to

GUJARAT TECHNOLOGICAL

UNIVERSITY

In consideration of being a Research Scholar at Gujarat Technological University, and in the interests of the facilitation of research at the University and elsewhere, I, **Patel Dhruvisha Pravinbhai** having Enrollment No. **229999906001** hereby grant a non-exclusive, royalty free and perpetual license to the University on the following terms:

- a. The University is permitted to archive, reproduce and distribute my thesis, in whole or in part, and/or my abstract, in whole or in part (referred to collectively as the "Work") anywhere in the world, for non-commercial purposes, in all forms of media;
- b. The University is permitted to authorize, sub-lease, sub-contract or procure any of the acts mentioned in paragraph (a);
- c. The University is authorized to submit the Work at any National / International Library, under the authority of their "Thesis Non-Exclusive License";
- d. The Universal Copyright Notice (©) shall appear on all copies made under the authority of this license;
- e. I undertake to submit my thesis, through my University, to any Library and Archives. Any abstract submitted with the thesis will be considered to form part of the thesis.
- f. I represent that my thesis is my original work, does not infringe any rights of others, including privacy rights, and that I have the right to make the grant conferred by this non-exclusive license.
- g. If third party copyrighted material was included in my thesis for which, under the terms of the Copyright Act, written permission from the copyright owners is required, I have obtained such permission from the copyright owners to do the acts mentioned in paragraph (a) above for the full term of copyright protection.
- h. I understand that the responsibility for the matter as mentioned in the paragraph (g) rests with the authors / me solely. In no case shall GTU have any liability for any acts / omissions / errors / copyright infringement from the publication of the said thesis or otherwise.
- i. I retain copyright ownership and moral rights in my thesis, and may deal with the copyright in my thesis, in any way consistent with rights granted

- by me to my University in this non-exclusive license.
- j. GTU logo shall not be used /printed in the book (in any manner whatsoever) being published or any promotional or marketing materials or any such similar documents.
 - k. The following statement shall be included appropriately and displayed prominently in the book or any material being published anywhere: "The content of the published work is part of the thesis submitted in partial fulfilment for the award of the degree of Ph.D. in **Science-Mathematics** of the Gujarat Technological University".
 - l. I further promise to inform any person to whom I may hereafter assign or license my copyright in my thesis of the rights granted by me to my University in this nonexclusive license. I shall keep GTU indemnified from any and all claims from the Publisher(s) or any third parties at all times resulting or arising from the publishing or use or intended use of the book / such similar document or its contents.
 - m. I am aware of and agree to accept the conditions and regulations of Ph.D. including all policy matters related to authorship and plagiarism.

Date: 09/10/2026

Place: Ahmedabad

D.P. Patel

Signature of the Research Scholar

Patel Dhruvisha Pravinbhai

Recommendation of the Research Supervisor:

Recommended  9/10/2026

Signature of the Research Supervisor

Dr. Brijesh Kumar A. Tripathi

Thesis Approval Form

The viva-voce of the PhD Thesis submitted by **Patel Dhruvisha Pravinbhai** (Enrollment No. 229999906001) entitled **Differential Geometry of Finsler Space with Special Metrics** was conducted on **Friday, 09/10/2026** (day and date) at Gujarat Technological University.

(Please tick any one of the following options)

- ☒ The performance of the candidate was satisfactory. We recommend that he/she be awarded the PhD degree.
- ☐ Any further modifications in research work recommended by the panel after 3 months from the date of first viva-voce upon request of the Supervisor or request of Independent Research Scholar after which viva-voce can be re-conducted by the same panel again.

(briefly specify the modifications suggested by the panel)

- ☐ The performance of the candidate was unsatisfactory. We recommend that he/she should not be awarded the PhD degree.

(The panel must give justifications for rejecting the research work)

Name and Signature of Supervisor with

Seal **Dr. Boijesh Kumar**
A. T. Chopra

ASSISTANT PROFESSOR IN MATHEMATICS
SCIENCE & HUMANITIES DEPARTMENT
L.D. COLLEGE OF ENGINEERING
NAVRANGPURA, AHMEDABAD

1) (External Examiner 1) Name and

Signature **Prof. Dr. Prakash**

2) (External Examiner 2) Name and

Signature **Prof. Bankteshwar Tiwari**

ABSTRACT

In the realm of differential geometry, Finsler geometry extends Riemannian concepts by allowing the metric to be more general than the quadratic form. Finsler geometry is just Riemannian geometry without quadratic restriction, commented by the famous geometer, S. S. Chern. In fact, as early as 1854, B. Riemann had introduced the concept of Finsler geometry in his ground-breaking Habilitationsvortrag. No essential development was made until 1918 when P. Finsler studied the geometry of curves and surfaces in the general case. A particularly important class of Finsler metrics is the generalized (α, β) -metrics, which depend on a Riemannian metric α and a 1-form β . These metrics offer flexibility in modeling geometrical structures and physical phenomena.

The present thesis, entitled DIFFERENTIAL GEOMETRY OF FINSLER SPACE WITH SPECIAL METRICS, has been divided into **ten** chapters.

Chapter 1 establishes the theoretical and historical framework of this thesis by tracing the structural development of Finsler geometry as a natural generalization of Riemannian geometry. We begin with a concise overview of the transition from isotropic to anisotropic spaces, emphasizing the pivotal role of the Finsler metric function $F(x, y)$. The chapter introduces the fundamental definitions and key concepts of Finsler geometry, along with a discussion of the various types of Finsler metrics employed throughout this work. Furthermore, a comprehensive literature survey is presented, followed by a discussion of the known applications of Finsler geometry and an outline of its future scope and promising directions for further research.

Chapter 2 is devoted to the characterization of projective changes between specific classes of (α, β) -metrics. We establish necessary and sufficient conditions for a projective transformation between a special cubic (α, β) -metric, defined by $F = \frac{(\alpha + \beta)^3}{\alpha^2}$, and a Kropina metric $\tilde{F} = \frac{\alpha^2}{\beta}$. The same analytical framework is then applied to investigate the projective transformation between the square root (α, β) -metric $F = \sqrt{\alpha(\alpha + \beta)}$ and a Kropina

metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$. Beyond characterizing these projective mappings, we derive the fundamental conditions required for F to be projectively related to $\tilde{F}$. A substantial portion of this analysis is dedicated to the study of curvature symmetries, wherein we explicitly determine the conditions under which F admits isotropic S -curvature and isotropic Berwald curvature.

Chapter 3 introduces a new class of Finsler metric, namely the exponential (α, β) -metric $F = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$, and investigates its geometric structure in detail. We establish the necessary and sufficient conditions under which this metric determines a Berwald space, and further derive the conditions required for it to be classified as a Douglas space. Furthermore, several special cases are examined to determine the precise conditions under which the metric reduces to a weakly Berwald space.

Chapter 4 investigates the curvature properties of two significant Finslerian structures, namely the cubic (α, β) -metric and the generalized Matsumoto metric. The chapter is devoted to the study of locally projectively flat structures under the assumption of constant Ricci curvature. In particular, we derive the conditions under which these metrics admit vanishing flag curvature ($K = 0$).

Chapter 5 investigates the locally dually flat and projectively flat properties of the cubic (α, β) -metric and the exponential (α, β) -metric in the presence of isotropic S -curvature. We establish the necessary and sufficient conditions for these metrics to exhibit dual flatness. Furthermore, rigorous proofs are presented for the conditions under which these metrics reduce to locally Minkowskian spaces. These findings contribute significantly to the classification of special (α, β) -metrics and their associated curvature invariants.

Chapter 6 presents the development of a broader class of Finsler metrics extending beyond the traditional (α, β) -framework. These metrics are constructed from a Riemannian metric α together with two differential 1-forms, $\beta = b_i(x)y^i$ and $\gamma = d_i(x)y^i$. We derive the

spray coefficients and the Cartan connection to elucidate the intrinsic geometry of these non-standard structures. By establishing the necessary and sufficient conditions for projective correspondence, we demonstrate that this broader class yields numerous novel Douglas metrics. This work extends the scope of Finsler geometry to multi-form manifolds, thereby providing new tools for the study of geodesic-preserving transformations.

Chapter 7 investigates the S -curvature and mean Berwald curvature for a class of homogeneous exponential (α, β) -metric. Since the curvature of homogeneous spaces is determined by the algebraic structure of the underlying Lie algebra, we derive an explicit formula quantifying the rate of change of the distortion along geodesics. Furthermore, these results are employed to provide a complete characterization of the mean Berwald curvature.

Chapter 8 investigates the emerging framework of $\mathbb{R}$ -complex Finsler spaces, a generalization of complex Finsler geometry in which the scalar field is restricted to $\lambda \in \mathbb{R}$. We rigorously derive the fundamental metric tensors, their determinants, and their inverses for a specialized cubic (α, β) -metric within this framework. Furthermore, we establish the necessary and sufficient conditions under which these metrics exhibit non-Hermitian characteristics, identifying the precise point at which standard complex symmetries break down. By presenting specific Riemannian examples of Hermitian cubic (α, β) -metric structures, this chapter provides a complete analytical profile of the interplay between complex structures and cubic (α, β) -metric deformations.

Chapter 9 introduces a new class of Finsler metrics arising from a Kropina-Randers type modification. The construction is based on a Riemannian metric $\rho = \sqrt{a_{ij}(x)y^i y^j}$ together with two differential 1-forms, $\beta = b_i(x)y^i$ and $\eta = \eta_i(x)y^i$. The resulting structure yields a broader family of metrics that generalizes the traditional (ρ, β) -metric framework. We investigate the geometric properties of this class and establish the necessary and sufficient conditions under which these metrics are locally projectively flat. Furthermore, explicit criteria are derived characterizing when these metrics belong to the Douglas type. These

results contribute to a deeper understanding of projective geometry in the context of Finsler spaces.

Chapter 10 presents a comprehensive summary and conclusion, synthesizing the key findings obtained throughout this thesis. The chapter encapsulates the progress achieved in the differential geometry of Finsler spaces equipped with special metrics, highlighting the significance of the newly introduced metric classes and their geometric properties. A detailed discussion is provided on the implications of the projective transformations, curvature conditions, and metric classifications established in this work. Finally, promising directions for future research are identified, along with potential extensions and applications that may further advance the theory and broaden the scope of Finsler geometry in both theoretical and applied domains.

Acknowledgement

Every achievement is backed by the unseen strength of Almighty God. Success is ultimately characterized as the eternal state of achievable perfection in all pursuits by those who persist, aided by the support of their predecessors, educators, family, friends, and colleagues. The successful completion of any research work is a matter of great endurance, but because of the help, cooperation, and support by many associated directly or indirectly makes the job enjoyable and simple. I consider it a privilege to express my sincere gratitude to all those individuals whose direct or indirect support and cooperation made the completion of my Ph.D. thesis possible.

The pathway to my Ph.D. has been truly enriched under the invaluable guidance of my respected supervisor, **Dr. Brijesh Kumar A. Tripathi**, Assistant Professor, Mathematics, Science and Humanities Department, L. D. College of Engineering, Ahmedabad, Gujarat. I take this opportunity to express my sincere and heartfelt gratitude to **Dr. Brijesh Kumar A. Tripathi**, without whom this work would not have been possible. His vast knowledge, remarkable patience, and exceptional way of explaining even the most complex concepts are beyond what words can truly express. His guidance has not only helped me academically but has also shaped my thinking and approach towards research. He has always been generous with his time, never hesitating to support and guide whenever I faced difficulties. No matter how busy he was, he was always available to listen, help, and provide direction. I feel extremely fortunate and honored to have worked under his supervision. I will always remain deeply grateful for his guidance, support, and blessings. Thank you, sir.

I would like to express my sincere gratitude to the members of my Doctoral Program Committee (DPC), **Dr. Manoj Kumar Singh**, Assistant Professor, Government Engineering College, Dahod, and **Dr. Kunjankumar Shah**, Assistant Professor, Indian Institute of Teacher Education (IITE), Gandhinagar. Their expert guidance, clear explanations, and valuable suggestions have been of immense help throughout the course of my research. I am deeply thankful for their continuous support, thoughtful feedback, and for always pro-

viding the best possible direction at every stage of this work.

I am deeply grateful to **Prof. Sanjay K. Patel**, Head of the Science and Humanities Department, L. D. College of Engineering, Ahmedabad, Gujarat, India, along with all the respected faculty members and staff of the institute. Their constant support, encouragement, and the enriching academic environment they have created have significantly contributed to my intellectual and professional development.

Furthermore, I express my appreciation to the staff of the **Ph.D. Section** at Gujarat Technological University, Gujarat, India for their assistance and cooperation in ensuring the smooth progress of my research. Their assistance in managing the documentation, DPC formalities, and various procedural aspects was seamless, ensuring that I could focus entirely on my research objectives. I acknowledge with gratitude the financial support received from the **Government of Gujarat** under the **SHODH Scheme** of Developing High-quality Research. This scheme provided essential financial support throughout this journey, allowing me to focus entirely on my research goals.

A special thanks to my senior, **Dr. Sadikabanu Musirahmed Khan**, for her constant support and guidance throughout this journey. Without her encouragement and help, this journey would have been much more difficult. I am truly grateful for the way she treated me like a sister and always stood by me whenever I needed support. Her kindness and motivation have made this journey smoother and more meaningful. I feel incredibly fortunate to have had a Ph.D. colleague like **Sejal Prajapati**, who has been a pillar of my research journey. Her constant motivation and unwavering support were the foundation of my progress. I am deeply grateful to her for always standing by me, through both the technical challenges and the personal milestones of this endeavor. My heartfelt thanks also go to **Ankita Sinha and Krupal Patel** for their constant motivation; this journey would have been far more difficult without their unwavering support and friendship.

Finally, I owe my deepest gratitude to my parents, **Mr. Pravinbhai Patel** and **Mrs. Rasilaben Patel**. Without their sacrifices and blessings, nothing would have been possible. Thank you for your unconditional love and the constant support which I have always needed and leaned upon. This achievement is a tribute to your faith in me. I also extend my heartfelt thanks to my little brother, **Deep Patel**; though he is younger, he has always supported me like a ‘Big Brother’ and motivated me to keep going. Thank you for being the best brother. This journey was made possible only through the constant love and support of my family, which has been my greatest strength throughout. My sincere gratitude also goes to my cousin, **Bhumi Patel**. I am fortunate to have a sister who understood the demands of my doctoral work and consistently stood by my side.

My journey would not have been the same without the people who walked it with me. To everyone who offered a word of encouragement or a helping hand, thank you. I am profoundly blessed to have had your support and guidance throughout this Ph.D. endeavor.

Patel Dhruvisha Pravinbhai

*This Thesis is lovingly dedicated to
my parents, whose unconditional love
and sacrifices made this possible,

and to my brother,
my forever source of encouragement.*

Contents

Abstract	xii
Acknowledgement	xvi
List of Symbols	xxiii
1 Introduction	1
1.1 A Concise Overview of Geometry	1
1.2 Structural Development in Finsler Geometry	2
1.3 Fundamental Concepts of Finsler Geometry	7
1.4 Finsler Metric	15
1.5 Literature Survey	21
1.6 Scope and Applications of Finsler Geometry	24
2 On a Special Finsler (α, β)-Metric	27
2.1 Introduction	27
2.1.1 Preliminaries	28
2.1.2 Projective Transition of the Two Metrics	31
2.1.3 Projective Change with Curvature Properties	39
2.2 Projective Transformation Between Square root (α, β) -Metric and Kropina Metric	43
2.2.1 Projective Change with Curvature Properties	51
3 Finser Space with (α, β)-Metric	55
3.1 Introduction	55
3.1.1 Preliminaries	56
3.1.2 The Condition to be a Berwald space	60
3.1.3 The Condition to be a Douglas Space	62
3.2 The Condition to be a Weakly-Berwald Space	70
4 Projectively Flat of Finsler-Metric with Constant Flag Curvature	81
4.1 Introduction	81
4.1.1 Preliminaries	82
4.1.2 Equations of Geodesics of Projectively Flat Cubic (α, β) -Metric .	84

4.1.3	Projectively Flat Cubic (α, β) -Metric with $K = 0$	87
4.2	Geometric Equations for the Projectively Flat Generalized Matsumoto Metric	91
4.2.1	Projectively Flat Generalized Matsumoto Metric with $K = 0$. . .	94
5	Locally Dually Flatness Properties in Finsler (α, β)-Metrics	97
5.1	Introduction	97
5.1.1	Preliminaries	98
5.1.2	Characterization of Locally Dually Flat Cubic (α, β) - Metric . . .	100
5.2	Characterization of Locally Dually Flat Exponential (α, β) -Metric . . .	105
6	A New Class of Finsler Metrics	112
6.1	Introduction	112
6.2	Preliminaries	115
6.3	Matsumoto type (F, β) -metrics	116
6.4	Spray Coefficients of Matsumoto type (F, β) -Metrics	121
6.5	Cartan Connection of Matsumoto type (F, β) -Metrics	126
6.6	Proof of Theorem 6.1.1	132
6.6.1	Proof of Corollary 6.1.2	133
6.6.2	Proof of Theorem 6.1.3	134
7	Curvatures on Homogeneous Finsler Space with Finsler Metrics	135
7.1	S-curvature Homogeneous Exponential (α, β) -Metric	135
7.2	Mean Berwald Curvature	140
8	On $\mathbb{R}$-Complex Finsler Space with Special Finsler Metrics	150
8.1	Introduction	150
8.2	Preliminaries	151
8.3	$\mathbb{R}$ -Complex Cubic (α, β) -Metric	153
8.4	Fundamental Tensor of $\mathbb{R}$ -Complex Finsler Space with Cubic (α, β) -Metric	157
9	On (ρ, β, η)-Metrics with Kropina - Rander Changes	172
9.1	Introduction	172
9.2	Preliminaries	173
9.3	(ρ, β, η) - Metrics for a Kropina-Randers Change	175

9.4	Spray Coefficients of F	182
9.5	Projectively Flat (ρ, β, η) -Metrics	187
9.6	Douglas Spaces by (ρ, β, η) -Metrics	190
10	Conclusion and Future Scope	193
10.1	Conclusion	193
10.2	Future Scope	195
	References	197
	List of Publications	212

List of Symbols

1. M : Differentiable manifold of dimension n
2. $T_p M$: Tangent space to M at the point p
3. p : A point on the manifold M
4. (x, y) : Local coordinates on the tangent bundle TM , where $x = (x^i)$ denotes the position and $y = (y^i)$ denotes the tangent vector
5. $F(x, y)$: Finsler metric function on M
6. $F^n = (M, F)$: An n -dimensional Finsler space
7. $\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$: Riemannian metric baseline defined on M
8. $\beta(x, y) = b_i(x)y^i$: Differential one-form defined on M
9. $g_{ij}(x, y)$: Fundamental Finsler metric tensor
10. $G^i(x, y)$: Geodesic spray coefficients of the Finsler space
11. G_{jk}^i : Coefficients of the Berwald connection, defined as $G_{jk}^i = \partial_j \partial_k G^i$
12. $\mathbb{R}^n$: n -dimensional real vector space
13. $\mathbb{R}$ -Complex Finsler space : Real-complex Finsler space
14. C_{ijk} : Cartan torsion tensor, defined as $C_{ijk} = \frac{1}{2} \dot{\partial}_k g_{ij}$
15. $S(x, y)$: S -curvature of the Finsler space
16. D_{jkl}^i : Douglas curvature tensor
17. ∇ : Covariant derivative operator
18. $K(P, y)$: Flag curvature of the Finsler space along flagpole y and flag P
19. $\text{Ric}(y)$: Ricci scalar of the Finsler space, defined by $\text{Ric}(y) = R_m^m(x, y)$
20. $E_{ij}(x, y)$: Mean Berwald curvature tensor

CHAPTER 1

Introduction

1.1 A Concise Overview of Geometry

In ancient times, the field of mathematics was primarily categorized into two core disciplines: geometry and the study of numbers. Geometry is a discipline of mathematics that studies the mathematical description, classification, and relationships between shapes. At its essence, geometry defines the relationships between two points. Conversely, differential geometry utilizes Differential Calculus and Integral Calculus as methods to explore geometric problems.

Geometry is the study of relationships between points, lines, surfaces, solids, and higher dimensions. It is arguably one of the most ancient areas of mathematics. The discipline emerged from the practical necessity of land measurement and further evolved through its applications in architecture and the arts.

Geometry (Greek *γεωμετρία*; geo = earth, metria = measure) arose as the field of knowledge that deals with spatial relationships. It is defined as:

“The science that treats of the shape and size of things”.

Plato, the great philosopher, was so interested in the subject that he had the following words carved on the front entrance of his Platonic school, the Academy.

“Let no man ignorant of geometry enter”.

1.2 Structural Development in Finsler Geometry

The constant progression of geometry has long been a fascinating subject. The term geometry comes from the Greek terms for earth measure. The Babylonians and Egyptians used to measure the length and width of rectangular fields for taxation purposes, according to historical documents. The field originated circa 4000BC. Egypt's pyramids, built around 3000BC, showcase early engineering skills that likely utilized geometric principles. Greek mathematicians not only devised empirical formulas for areas and volumes, but also used line segments, polygons, circles, and other conics to represent numbers. In the early 17th century, Kepler and Galileo made substantial advances to geometry and astronomy. Cavalieri, Galileo's student, believed that moving points could generate lines and moving lines could generate planes. Gauss, in the early 19th century, considered geometries that did not follow Euclid's fifth postulate, also known as the parallel postulate. Riemann extended Gauss's curvature theory, which served as the foundation for relativity theory. During the 19th century, distance functions and metrics were developed further by Cayley and Klein. Later, these theories were expanded to 3 and n -dimensions.

The exploration of spaces equipped with generalized metrics began with P. Finsler in 1918. It is typically viewed as an extension of Riemannian geometry. In 1854, B. Riemann suggested studying more general metric spaces beyond Riemannian geometry in his lecture. However, he himself doubted that the geometric meanings of quantities in such generalized spaces were clear enough to contribute meaningfully to geometry. As a result, there was a 60-year neglect in studying this geometry. Finsler began studying geometry by analyzing problems in the calculus of variations. J. Synge, V. Wagner, L. Berwald, E. Cartan, H. Rund, M. Matsumoto, and other researchers established Finsler geometry as a distinct branch of differential geometry. In contemporary applications, classical Finsler geometry describes the geometry of vector fiber bundles on manifolds. Finsler geometry can be seen as a form of Riemannian geometry that does not adhere to the quadratic constraint. It also finds significant applications, particularly in the theory of relativity, control theory, and mathematical biology.

The evolution of Finsler geometry can be categorized into four distinct phases.

1. 1st–Period (1924–1933)
2. 2nd–Period (1934–1950)
3. 3rd–Period (1951–1963)
4. 4th–Period 1963–up to now

1st–Period (1924–1933)

In 1925, J. H. Taylor [146], J. L. Synge [140], and L. Berwald [21] began their work in Finsler geometry. Berwald was the pioneer in introducing the idea of connection within Finsler space theory. In 1928, Taylor gave the name Finsler space to a manifold equipped with generalized metric.

In his 1854 habilitation lecture, B. Riemann introduced the concept of n -dimensional differential manifolds and proposed studying manifolds using a Riemannian metric.

$$ds^2 = g_{ij}(x)dx^i dx^j \quad i, j = 1, 2, 3, \dots n.$$

Before arriving at the above metric he had also considered the metric

$$ds^3 = g_{ijk}(x)dx^i dx^j dx^k.$$

And

$$ds^4 = g_{ijkl}(x)dx^i dx^j dx^k dx^l.$$

Where ds represents the distance between two adjacent points x and $x + dx$ in the manifold.

P. Finsler's study was inspired by his instructor supervisor Caratheodory's notion to geometrize variation calculus. The entirety of his thesis is composed of generalizations of

traditional findings in the study of curves and surfaces within a standard space. In the preface of his thesis, he stated:

“A treatment of geometry under the most general possible assumptions has a metric to let us know to what extent every theorem depends on special assumption of the merit or the number of dimensions.”

Finsler had already introduced a metric tensor

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$$

and the C -tensor

$$C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}$$

which are the well-known basic tensors of Finsler geometry, first presented in his thesis. Actually, Finsler has confined himself to only those tensors that are zero in Riemannian geometry. Finsler served as a professor of mathematical logic at the University of Zurich. He was more of a philosopher foundationalist than a mathematician.

2nd-Period (1934-1950)

The 2nd phase commenced in 1934 when E. Cartan [26] published his monograph on Finsler geometry. In 1935, E. Cartan came out with a definite establishment of Finsler geometry. Despite this, several issues remained unresolved. In brief notes, Cartan revealed his set of axioms to determine a connection in a Finsler space. It was fully published in his well-known monograph in 1936.

Consider a curve C , $x^i = x^i(t)$ with a parameter t of M^n . Then we have a system of differential equations

$$\frac{dp}{dt} = \frac{dx^i}{dt} e_i, \quad \frac{de_i}{dt} = \Gamma_{ij}^k(x(t)) \frac{dx^j}{dt} e_k,$$

where Γ_{ij}^k are functions *a priori* given in every coordinate neighborhood of M .

Furthermore, if a manifold M with the affine connection $\Gamma = \{\Gamma_{ij}^k(x)\}$ is equipped with a Riemannian metric $ds^2 = g_{ij}(x)dx^i dx^j$, and the moving frame field $\{e_i(t)\}$ satisfies the compatibility condition $g_{ij} = e_i \cdot e_j$ along any curve, then $\{\Gamma_{ij}^k\}$ is called a metric connection. The condition for $\{\Gamma_{ij}^k\}$ to be compatible with the metric is given by:

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ijk} + \Gamma_{jik}, \quad \text{where } \Gamma_{ijk} = g_{ir}\Gamma_{jk}^r.$$

By cyclically permuting the indices, the relationship between this connection and the unique torsion-free Levi-Civita connection γ_{ijk} becomes:

$$\gamma_{ijk} = g_{ir}\gamma_{jk}^r = \Gamma_{ijk} - \frac{1}{2}(T_{ijk} - T_{jki} + T_{kij}).$$

Therefore, if we assume the connection is symmetric (torsion-free), the torsion tensor vanishes ($T_{ijk} = 0$), which implies that the connection reduces directly to the standard Riemannian Christoffel symbols:

$$\Gamma_{ijk} = \gamma_{ijk}.$$

Now, the central problem is how to generalize the moving frame field $\{e_i\}$ to a Finsler space. To resolve this issue, Cartan introduced an entirely new framework: the line element space $\bar{m}$ of an n -dimensional Finsler space M . Instead of working purely on the base manifold, he anchored the connection to a coordinate pair (x^i, y^i) , consisting of a position point x^i and a directional vector y^i . Cartan termed this pair the supporting element, redefining $\bar{m}$ as the geometric space composed of all such supporting elements.

3rd-Period (1951–1963)

The third phase in the development of Finsler geometry occurred in 1951 when H. Rund [111] introduced a new process of parallelism from the standpoint of Minkowskian geometry. Several mathematicians such as W. Barthel [16], A. Diecke [43], D. Laugwitz [72],

and R. Sulanke [139] studied Finsler spaces using Rund's approach.

D. Laugwitz published many papers in 1954; his early subject was the theory of connections, and he was especially interested in Minkowski geometry. His pupil E. Heil mainly studied the theory of convex bodies, but also wrote various interesting papers on Finsler geometry.

Moor published many papers since 1950. In 1957, he treated three-dimensional Finsler spaces by referring to a natural frame field which was introduced by generalizing Berwald's idea to the three-dimensional case.

We must not forget the pioneering work of R. S. Ingarden [106] on Finsler geometry from the standpoint of application. His monograph (1957), originally his thesis, gave a very interesting geometrical theory of the problem of electron microscopy. In his paper we find a truly surprising theorem:

“Any n -dimensional Finsler space can be isometrically embedded in a $2n$ -dimensional Minkowski space.”

4th-Period (1963–present)

The 4th phase in the evolution of Finsler geometry commenced in 1963 when H. Akbar Zadeh [4] established the contemporary theory of Finsler spaces, rooted in the geometric principles of connections in fiber bundles. The goal of this development was to discover new models of Finsler spaces, and the contribution of Professor M. Matsumoto regarding the advancement of Finsler geometry deserves recognition. He linked the Cartan connections, Berwald connections, and Rund connections using methods known as the C -process and the P -process. His research publications on Finsler space theory with (α, β) -metric, published between 1992 and 1996, made significant contributions to the development of Finsler spaces. Several Indian mathematicians made notable contributions during this period, including Prof. U. P. Singh, Prof. B. Tiwari, Prof. Pradeep Kumar, Prof. H. D. Pandey, Prof. R. B. Mishra, Prof. Gauri Shanker, Prof. M. D. Upadhyay, Prof. R. S. Sinha, Prof. B. B. Sinha, Prof. Ram Hit, Dr. B. N. Prasad, Prof. T. N. Pandey, Prof. H. S. Shukla, Prof. P. N. Pandey, Prof. V. K. Chaubey, Prof. S. K. Narasimhamurthy, and Prof. C. S.

Bhagwati, among others.

1.3 Fundamental Concepts of Finsler Geometry

1. Manifold

The manifold is a topological space that seems locally to resemble a Euclidean space point. More precisely, each point of an n -dimensional manifold has a neighborhood that is homeomorphic to the Euclidean space of dimension n . In these more precise forms, the manifold is read as an n -manifold. 1-D manifolds comprise circles and lines. Two-dimensional (2D) manifolds are described as surfaces. The models include the torus, sphere, and plane, all of which can be embedded (formed without self-joints) in 3-D real-space, with the exception of the Klein bottle and real projective flat, which can self-cross once immersed in 3-D (three-dimensional) real-space. Manifolds serve various purposes in the design of PCs and enhance realism, as they are essential for connecting images (surfaces) to facilities (e.g., CT scans). In an augmented reality context, an image (digression plane) can be seen as something associated with a coordinate system, and by using tensors to track movement and rotation, one can understand the image's position and orientation in space.

2. Curve and Line Element

Let R be a region of n -dimensional differentiable manifold M^n which is covered completely by a coordinate system such that any point P of R is represented by a set of n real independent variables $x^i (i = 1, 2, 3, \dots, n)$ called the coordinates of the point. A coordinate transformation is represented by a set of n -equations [97],

$$x^{i'} = x^{i'}(x^1, x^2, \dots, x^n), \quad (i' = 1, 2, 3, \dots, n), \quad (1.3.1)$$

which shows that the coordinates x^i of the point P of M^n are represented in the new coordinate system of new variables $x^{i'}$. We assume that the functions $x^{i'}$ of (1.3.1)

are at least of class C^2 and [97]

$$\det \left(\frac{dx^{i'}}{dx^i} \right) \neq 0. \quad (1.3.2)$$

A collection of R points whose coordinates can be written as a function of a single parameter t is considered a curve of M^n , culminating in the equations,

$$x^i = x^i(t), \quad (1.3.3)$$

which describes a curve C of M^n . If the function (1.3.3) is of class C^1 , we consider the entity, whose components are given by,

$$y^i = \frac{dx^i}{dt}, \quad (1.3.4)$$

as the tangent vector to C . We call the combination (x^i, y^i) a line element of C .

3. Finsler Space

Let the function $F(x^i, y^i)$ be defined for all the line elements in the region R and be of class C^5 in all the $2n$ arguments. If we define the infinitesimal distance ds between two points $P(x^i)$ and $Q(x^i + dx^i)$ of R by the relation [97]:

$$ds = F(x^i, dx^i) \quad (1.3.5)$$

then the space X_n equipped with the essential function F defining the metric (1.3.5) is called a Finsler space, if $F(x^i, dx^i)$ satisfies the following conditions:

Condition (A1):

The function $F(x^i, y^i)$ is positively homogeneous of degree one in y^i i.e.,

$$F(x^i, py^i) = pF(x^i, y^i), \quad p > 0. \quad (1.3.6)$$

The necessary and sufficient condition for the integral to be independent of the choice of parameter is known as the arc-length. It has been shown that the choice of parameter t has no effect on $\int F(x, y) dt$.

Condition (B1):

The function $F(x^i, y^i)$ is positive if not all y^i vanish simultaneously, that is,

$$F(x^i, y^i) > 0 \text{ with } \sum_i (y^i)^2 > 0. \quad (1.3.7)$$

Thus, the distance between two distinct points is positive.

Condition (C1):

The quadratic form,

$$\dot{\partial}_i \dot{\partial}_j F^2(x^i, y^i) \xi^i \xi^j = \frac{\partial^2 F^2(x^i, y^i)}{\partial y^i \partial y^j} \xi^i \xi^j, \quad (1.3.8)$$

is assumed to be positive definite for all any variable ξ^i . More precisely, $F(x^i, y^i)$ is a convex function in y^i .

In this manner a metric is imposed on our X_n . The space X_n is called a Finsler space if $F(x^i, y^i)$ defining the metric (1.3.5) satisfies the condition (A1), (B1), and (C1).

4. Tangent Space

We examine a transformation of local coordinates expressed by equation (1.3.1), along the curve (1.3.3) with respect to an invariant parameter t ; the components of

the tangent vector $y^{i'} = \frac{dx^{i'}}{dt}$ are derived by differentiating this relationship [97].

$$x^{i'} = x^{i'}(x^i(t)), \quad (1.3.9)$$

with respect to t , this gives,

$$y^{i'} = \frac{\partial x^{i'}}{\partial x^i} y^i, \quad (1.3.10)$$

or in terms of differentials,

$$dx^{i'} = \frac{\partial x^{i'}}{\partial x^i} dx^i. \quad (1.3.11)$$

Here dx^i is interpreted as the components of a displacements in M^n from the point $P(x^i)$ to a neighbouring point $Q(x^i + dx^i)$.

If the point $P(x^i)$ is fixed, i.e., the coefficients $\frac{\partial x^{i'}}{\partial x^i}$ of the transformation (1.3.11) are fixed, this relation represents a linear transformation of the dx^i onto $dx^{i'}$. The same is true for the variables y^i and $y^{i'}$ in the transformation (1.3.10). Therefore, the n quantities of this kind may be taken to define the elements of an n -dimensional linear vector space [97].

A system of n -quantities x^i whose transformation law under (1.3.1) is equivalent to that of the y^i is called n contravariant vector attached to the point $P(x^i)$ of M^n . Such contravariant vectors constitute the elements of a vector space.

5. Metric Tensor

From equation (1.3.11), we can easily see that the quantities g_{ij} defined by $g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$ form the components of the covariant tensor of rank 2. Also, $g_{ij}(x, y)$ are positively homogeneous of degree zero in y^i and symmetric in their indices. Due to homogeneity, from condition A we have [97]

$$F^2(x, y) = g_{ij}(x, y) y^i y^j. \quad (1.3.12)$$

By condition C, it follows that inverse of the metric g_{ij} exists. Thus, if g^{ij} denotes the inverse of g_{ij} , then

$$g_{ij}(x, y)g^{jk}(x, y) = \delta_i^k, \quad (1.3.13)$$

where δ_i^k is well-known Kronecker delta. Therefore, the tensor whose covariant and contravariant components are $g_{ij}(x, y)$ and $g^{ij}(x, y)$ is called the metric tensor or the first fundamental tensor of the Finsler space F^n . The tensor $C_{ijk}(x, y)$ defined by

$$C_{ijk} = \frac{1}{2} \partial_k g_{ij} = \frac{1}{4} \partial_i \partial_j \partial_k F^2, \quad (1.3.14)$$

is positively homogeneous of degree-1 in y^i and is symmetric in all their indices. This tensor is called Cartan's C-tensor and satisfy [97]

$$C_{ijk}(x, y)y^i = C_{ijk}(x, y)y^j = C_{ijk}(x, y)y^k = 0. \quad (1.3.15)$$

$$(\partial_h C_{ijk})y^i = (\partial_h C_{ijk})y^j = (\partial_h C_{ijk})y^k = 0. \quad (1.3.16)$$

6. Curvature

In 1854, B. Riemann presented the concept of curvature for spaces equipped with a collection of inner products (Riemannian metrics). No notable advancements occurred in the broader context until 1918, when P. Finsler examined the variation problem in spaces characterized by a family of norms (Finsler metrics). At the same time, A. Einstein applied Riemannian geometry to formulate his theory of general relativity. By that point, however, the study of Finsler spaces was still in its early developmental phase. Up until 1926, L. Berwald extended Riemann's concept of curvature to Finsler spaces and introduced a novel non-Riemannian quantity through his connection.

Riemann curvature:

The Riemannian curvature $\mathbf{R}$ of F is a family of linear transformations $\mathbf{R}_y : T_x M \rightarrow T_x M$ which is defined by $\mathbf{R}_y = R_j^i dx^j \otimes \frac{\partial}{\partial x^i}$, where $y \in T_x M_0$ and

$$R_j^i = 2(G^i)_{xj} - y^k (G^i)_{x^k y^j} + 2G^k (G^i)_{y^k y^j} - (G^i)_{y^k} (G^k)_{y^j}. \quad (1.3.17)$$

The trace of the Riemann curvature R_j^i is called the **Ricci curvature** and is denoted by $Ric(x, y)$, where

$$Ric(x, y) = R_m^m(x, y).$$

If the Finsler metric has scalar flag curvature $K(x, y)$, then

$$Ric(x, y) = (n - 1)K(x, y)F^2. \quad (1.3.18)$$

Another important curvature in Finsler geometry is the flag curvature. The notion of flag curvature in Finsler geometry was also introduced by L. Berwald. One of the fundamental problems in Finsler geometry entails the study of Finsler metrics having constant flag curvature. The flag curvature, which generalizes the sectional curvature in Riemannian geometry does not depend on whether one is using Berwald, Chern, or Cartan connection [117].

Flag Curvature:

The flag curvature of a Finsler space (M, F) is the function $\mathbf{K} = \mathbf{K}(P, y)$ of a tangent plane $P = \text{span}\{y, u\} \subset T_x M$ and a vector $y \in P \setminus \{0\}$, which is defined by

$$\mathbf{K}(P, y) = \frac{g_y(\mathbf{R}_y(u), u)}{g_y(y, y)g_y(u, u) - [g_y(u, y)]^2}. \quad (1.3.19)$$

If $\mathbf{K}(P, y)$ is independent of P containing $y \in T_x M_0$, then we see that $\mathbf{K}(P, y) = \mathbf{K}(x, y)$ is a scalar function on $T_x M_0$. In this case, it is said that F has scalar flag curvature.

In the case that $\mathbf{K} = \text{constant}$, it is said that F has constant flag curvature [136].

In addition to flag curvature, there are several other essential quantities in Finsler

geometry. Here, we examine some of these important aspects.

6. Geodesics

In a Finsler space, geodesics represent the paths of shortest or longest arc-length connecting any two points within the space. The differential equations of a geodesic in a Finsler space are given by [97]:

$$\frac{d^2 x^i}{ds^2} + 2G^i \left(x, \frac{dx}{ds} \right) = 0, \quad (1.3.20)$$

where s is the arc-length parameter of the curve $x^i = x^i(s)$, and the spray coefficients are defined as:

$$2G^i = \gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds}, \quad (1.3.21)$$

where,

$$\gamma_{jk}^i = \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right).$$

8. Projective Change

Let us consider the quantities $G^i(x, y)$ appearing in the geodesic differential equations

$$\frac{d^2 x^i}{ds^2} + 2G^i(x, \frac{dx}{ds}) = 0, \quad (1.3.22)$$

where, $G^i = g^{ij}G_j$, $2G_j = \left(\frac{\partial^2 F}{\partial y^j \partial x^i} \right) y^i - \frac{\partial F}{\partial x^j}$ of a Finsler space.

$$\begin{cases} 2G_j = 2g_{ij}G^i = y^r \frac{\partial}{\partial x^r} \frac{\partial}{\partial y^j} \left(\frac{F^2}{2} \right) - \frac{\partial}{\partial x^j} \left(\frac{F^2}{2} \right) = \left(y^r \frac{\partial F}{\partial x^r} \right) \\ \frac{\partial F}{\partial y^j} + F y^r \frac{\partial}{\partial x^r} \frac{\partial F}{\partial y^j} - F \frac{\partial F}{\partial x^j}. \end{cases} \quad (1.3.23)$$

Above leads to define an operator g_i for a scalar field $S(x, y)$ by [10]

$$2g_i(S) = \left(y^r \frac{\partial S}{\partial x^r} \right) \frac{\partial S}{\partial y^i} + S y^r \frac{\partial^2 S}{\partial x^r \partial y^i} - S \frac{\partial S}{\partial x^i} \quad (1.3.24)$$

Taking into account ${}^*h_{ij} + {}^*l_i^*l_j = {}^*g_{ij}$ and ${}^*l_r^*g^{ir} = \frac{y^i}{{}^*F}$, the above expression may be rewritten in the form

$$2G^{*i} = 2G^i + \left(\frac{{}^*F|_0}{{}^*F} g^i - F^* g^{ij} K_j({}^*F) \right). \quad (1.3.25)$$

The differential equation of geodesics for any parameter t other than arc-length can be written as

$$\frac{d^2 x^i}{dt^2} + 2G^i(x, \frac{dx}{ds}) = \left(\frac{d^2 s}{dt^2} \right) \left(\frac{ds}{dt} \right) \frac{dx^i}{dt}. \quad (1.3.26)$$

Definition 1.3.1. A Finsler space $F^n = (M, F)$ coincides with a geodesic of ${}^*F^n(M, F)$ as a set of points and vice versa, then the transformation $F \rightarrow {}^*F$ is called a projective change and the two space F^n and ${}^*F^n$ are said to be projectively invariant.

Let us consider differential equation of geodesics of two projectively related Finsler spaces as

$$\frac{d^2 x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = \phi(t) \frac{dx^i}{dt}, \quad (1.3.27)$$

where $\phi(t) = \left(\frac{d^2 s}{dt^2} \right) \left(\frac{ds}{dt} \right)$ and $\frac{d^2 x^i}{dt^2} + 2{}^*G^i(x, \frac{dx}{dt}) = \phi(t) \frac{dx^i}{dt}$.

For two projectively related Finsler spaces, we have,

$$2\{G^i(x, \frac{dx}{dt}) - {}^*G^i(x, \frac{dx}{dt})\} = \{\phi(t) - \phi'(t)\} \frac{dx^i}{dt}. \quad (1.3.28)$$

The above can also be rewritten as

$${}^*G^i(x, y) = G^i(x, y) + P(x, y)y^i, \quad (1.3.29)$$

where, $P(x, y) = \frac{1}{2}\{\phi(t) - \phi'(t)\}$ and $y' = \frac{dx}{dt}$.

It is well established that a Finsler space $F^n = (M, F)$ is projective to a Finsler space ${}^*F^n = (M, {}^*F)$ if and only if ${}^*F$ satisfies one of the following three equations [10]:

1. ${}^*F_{(i)} - y^r \frac{\partial F_r}{\partial y^i} = 0,$
2. ${}^*l_i(j) - {}^*l_j(i) = 0,$
3. $\frac{\partial {}^*F_i}{\partial y^j} - \frac{\partial {}^*F_j}{\partial y^i} = 0.$

where $()$ is the h-covariant differentiation with respect to the Berwald connection of F^n .

1.4 Finsler Metric

Among the various classes of Finsler metrics, the (α, β) -metrics among the most important classes for theoretical and applied purposes. In this section, we explore some notable and significant Finsler metrics that we use in our research.

Definition 1.4.1. *Let M be a smooth manifold of dimension n . A **Finsler metric** on M is a function $F : TM \rightarrow [0, \infty)$ satisfying the following conditions:*

(F1) F is smooth on $TM_0 := TM \setminus \{0_x : x \in M\}$;

(F2) F is positively 1-homogeneous in the fiber direction, i.e.,

$$F(x, \lambda y) = \lambda F(x, y) \quad \forall \lambda > 0 \text{ and } (x, y) \in TM;$$

(F3) For each point $x \in M$, the fundamental tensor

$$g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}$$

is positive definite on TM_0 .

1. Randers Metric

In 1854, B. Riemann investigated metrics in their general forms and introduced the concept of curvature for a specific category of metrics known as Riemannian metrics [34]. This infinitesimal quantity accurately captures the local structure of Riemannian metrics and forms the foundation of Riemannian geometry. In 1918, P. Finsler examined the variational problem in manifolds characterized by a generalized Riemann metric. Subsequently, these metrics became known as Finsler metrics. Later, L. Berwald extended the concept of Riemann curvature to Finsler metrics by presenting what is known as the Berwald connection. Berwald also presented several non-Riemannian quantities through his connection. Since then, Finsler geometry has progressively advanced. Nevertheless, Finsler geometry is significantly more complex than Riemannian geometry. To understand the geometric interpretation of different quantities in Finsler geometry, one can start by examining the simplest type of non-Riemannian Finsler metrics, which are Randers metrics [34].

The Randers metric was first presented in 1941 by Norwegian -physicist, Randers [107] while he was engaged in research on general relativity. In the paper titled *On an Asymmetrical Metric in the Four-Space of General Relativity*, Randers introduced a simple asymmetrical extension of Riemannian metrics, stating, One of the most fundamental characteristics of the physical universe is the one-way nature of time-like intervals.

Randers metrics are essential and intrinsic Finsler metrics defined as a combination of a Riemannian metric and a differential 1-form. They originated from studies on general relativity and have found extensive applications across various fields of natural science, such as biology, physics, and psychology, among others. Specifically, Randers metrics can be naturally characterized as the solution to the Zermelo navigation problem. As a result, one can perform detailed calculations of different geometric quantities, allowing a deeper understanding

of the geometric properties of these metrics. Furthermore, Randers metrics possess significant non-Riemannian curvature characteristics. Analyzing Randers metrics will enhance our understanding of Finsler metrics.

The Randers metric represents the simplest case of a non-Riemannian Finsler metric on a smooth manifold M , constructed from a Riemannian metric and a differential 1-form defined by:

$$F = \alpha + \beta,$$

where $\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta(x, y) = b_i(x)y^i$ is a 1-form such that $\|\beta\| < 1$.

2. Kropina Metric

Kropina metrics were first introduced by L. Berwald in the context of a two-dimensional Finsler space featuring rectilinear extremals, and they were studied by V. K. Kropina [68]. This metric is physically significant as it characterizes the dynamical system described by a Lagrangian function, even though it exhibits a singularity [166]. Randers metrics are a type of regular Finsler metric, whereas Kropina metrics are Finsler metrics characterized by a singularity. Kropina metrics are among the simplest nontrivial Finsler metrics, offering various interesting applications in physics.

In 1961, Kropina examined the metric specifically identified in Berwald's article and determined the rectilinear extremals of their general structure. In 1973, Hashiguchi et al. [57] investigated two-dimensional Landsberg metrics and referred to them as the Kropina metric. The Kropina metric is characterized as follows:

$$F(x, y) = \frac{\alpha^2}{\beta}.$$

Furthermore, the Kropina metric was extended into a more general metric known

as the generalized m -Kropina metric. The Kropina and generalized m -Kropina metrics have numerous applications in fields such as physics, dissipative mechanics, electron optics in magnetic fields, and irreversible thermodynamics. A Finsler metric is referred to as a generalized m -Kropina metric when it is expressed in the form

$$F = \frac{\alpha^{m+1}}{\beta^m}, \quad (m \neq 0, -1),$$

where $\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta(x, y) = b_i(x)y^i$ is a differential 1-form metric.

3. Matsumoto-Metric

When contemplating an anisotropic medium in which the speed of light varies with direction, it is apparent that the unit surface is no longer a sphere; likewise, on Earth's surface, distances are sometimes measured in terms of time, such as the time indicated by a guidepost [103]. In this scenario, the measurement of a unit curve in minutes is converted into a general closed curve without a center. This phenomenon reflects the fact that traversing shorter distances uphill take less time than traversing equivalent lengths downward. As a result, the shortest route to a destination, such as a mountain peak, becomes a complex curve. Matsumoto established a precise model of a Finsler surface in 1989 and introduced a particular type of Finsler metric, as illustrated by [103]:

$$F = \frac{\alpha^2}{\alpha - \beta}. \quad (1.4.1)$$

Furthermore, several professors have investigated a generalized Matsumoto metric, defined as:

$$F = \frac{\alpha^{p+1}}{(\alpha - \beta)^p}, \quad (1.4.2)$$

where p is an integer with $p \neq 0, 1, -1$. This generalized form extends the

classical Matsumoto metric by introducing a parameter p , thereby enabling the consideration of a broader range of geometric structures and curvature behaviors. It allows one to explore how the interaction between the Riemannian metric α and the 1-form β influences the properties of Finsler spaces.

4. (α, β) -Metric

(α, β) -Finsler metrics represent a specific category of Finsler metrics constructed from a Riemannian metric α and a differential 1-form β . These metrics have attracted the attention of both Finsler geometers and physicists due to their elegant mathematical framework and practical applications in physics. Due to their greater flexibility compared to Riemannian metrics, they are well-suited for describing anisotropic spaces where direction dependent behavior is significant.

An (α, β) -metric on a connected smooth n -manifold M is a Finsler metric and can be represented as

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha}.$$

It is well known that $F = \alpha\phi(s)$ is a regular (α, β) -metric if the function $\phi(s)$ is a positive C^∞ function for $|s| < b_0$ satisfying the following condition:

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \leq b < b_0. \quad (1.4.3)$$

5. Cubic (α, β) -Metric

A generalized form of an (α, β) -metric on an n -dimensional manifold M defined as

$$F = \alpha \left(1 + \frac{\beta}{\alpha} \right)^p, \quad (1.4.4)$$

is known as the p -power (α, β) -metric [161], where $p \neq 0$ is a real constant. If $p = 1$, then equation (1.4.4) reduces to the Randers metric, which has sig-

nificant curvature properties and was first introduced by Ingarden in 1957. If $p = 2$, then it becomes the square metric, also known as Z. Shen's square metric. If $p = -1$, it reduces to the Matsumoto type metric, which can be used in the measurement of the slope of a mountain. If $p = \frac{1}{2}$, it reduces to the square root (α, β) -metric, i.e., $F = \sqrt{\alpha(\alpha + \beta)}$, and so on.

In the present work, we considered $p = 3$ in equation (1.4.4) and obtained a special class of (α, β) -metric of the form:

$$F = \frac{(\alpha + \beta)^3}{\alpha^2}, \quad (1.4.5)$$

named as the cubic (α, β) -metric on an n -dimensional manifold M . An n -dimensional Finsler space F^n equipped with the cubic (α, β) -metric is known as a Finsler space with cubic (α, β) -metric [150].

6. Exponential (α, β) -Metric

In the present investigation, we explore the fundamental properties of Finsler space and certain hypersurfaces using the exponential (α, β) -metric, expressed as:

$$F = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{\frac{-\beta}{\alpha}}. \quad (1.4.6)$$

7. (ρ, β, η) -Metric

The present work investigates a Kropina–Randers change within the framework of (ρ, β, η) -metric in Finsler geometry. This is motivated by the rich geometric implications of Randers-type deformations and their capacity to generate new Finsler structures from known ones. The Kropina–Randers change of a Finsler metric F is defined as follows:

$$\bar{F} = F + \eta,$$

where $F = \frac{\rho^2}{\beta}$ is the classical Kropina metric, $\rho = \sqrt{a_{ij}(x)y^i y^j}$ is a Rieman-

nian metric arising from the symmetric positive definite tensor $a_{ij}(x)$, and $\beta = b_i(x)y^i$, $\eta = \eta_i(x)y^i$ are two linearly independent differential 1-forms on an n -dimensional smooth manifold M . Substituting F into the above, the resulting metric takes the explicit form $\bar{F} = \frac{\rho^2}{\beta} + \eta$, which belongs to the general class of (ρ, β, η) -metric.

1.5 Literature Survey

Here is a brief overview of the research areas examined in this study.

In Finsler geometry, M. Matsumoto [85] introduced the concept of the (α, β) -metric which is an extension of the Randers metric. This concept has been utilized in Biology and Physics as an ecological metrics [11] and has been widely investigated by numerous scholars, who have formulated multiple (α, β) -metrics for instance, the polynomial (α, β) -metric, square metric, generalized Kropina metric, exponential (α, β) -metric, and Matsumoto metric centered on distinct geometric viewpoints.

In 1961, Rapcsák [105] studied the necessary and sufficient conditions for projective change, while in 1994, S. Bácsó and M. Matsumoto [15] studied the projective change between Finsler spaces with the (α, β) -metric. In 2008, H. S. Park and Y. Lee [100] investigated the projective change between a Finsler space with an (α, β) -metric and the corresponding Riemannian metric. Z. Shen and Civi Yildirim [125] investigated projectively flat metrics with constant flag curvature, also in 2008. G. Yang [161] introduced the notion of Einstein-reversibility for Finsler metrics. He studied a class of p -power Finsler metrics defined by a Riemannian metric and a 1-form, which are Einstein-reversible. It has been shown that if a p -power Finsler metric satisfies the Einstein-reversibility condition, then it must necessarily be an Einstein metric.

Berwald spaces were introduced by L. Berwald [21] in 1926, who characterized Finsler spaces whose connection coefficients depend only on position, i.e., with vanishing Berwald curvature. J. Douglas's work [47], which introduced projective in-

variants now known as Douglas curvature is where the idea of Douglas space first emerged. The notion of Douglas space was introduced by S. Bácsó and M. Matsumoto [17] as a generalization of Berwald space from the viewpoint of geodesic equations. In 2008, Y. Lee [73] defined a Douglas space of the second kind as a Finsler space with an (α, β) -metric. He also discovered that a Finsler space with a Matsumoto metric is a Douglas space of the second kind. Weakly Berwald space was defined by S. Bácsó and B. Szilágyi [12] in 2002 as a generalization of Berwald spaces. They also investigated the necessary and sufficient conditions for the existence of a weakly Berwald space of Kropina type. In 2023, B. K. Tripathi and S. Khan [151] examined a cubic (α, β) -metric, a special class of p -power Finsler metric, and derived the conditions under which a Finsler space with such a metric becomes a weakly Berwald space.

In 1980, Hashiguchi et al. [58] established the criterion for determining whether a Randers space is projectively flat. Z. Shen [132] discussed the classification problem of projective Finsler metrics of constant flag curvature. In addition to expressing a Taylor expansion or an algebraic formula, he constructed numerous examples that serve as models in Finsler geometry. In 2015, A. Tayebi and M. Nia [142] considered a class of Finsler metrics obtained by the Kropina change of the class of generalized m -th root Finsler metrics. They showed that a projectively flat Finsler metric with constant flag curvature is locally Minkowskian. They also found the necessary and sufficient conditions under which this class of metrics are locally dually flat.

The notion of locally dually flat Finsler metrics is fundamentally based on information geometry. In 2000, S.-I. Amari and H. Nagaoka investigated the principle of dual flatness in the setting of Riemannian metrics during their study on information geometry [9]. Later, Z. Shen extends the notion of dually flatness to Finsler metrics. In 2023, B. Tiwari and G. Prajapati [154] described locally dually flat (α, β) -metrics with generalized Matsumoto metric with isotropic S -curvature and scalar flag curvature, showing that these metrics reduce to locally Minkowskian metrics. S. K. Narasimhamurthy and D. M. Vasantha [95] investigated the locally dually flatness of

the special (α, β) -metric L , defined as $L^2(\alpha, \beta) = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2$, where c_i are constants.

T. Rajabi and N. Sadeghzadeh [106] introduced a new class of Finsler metrics which are not always (α, β) -metrics. They calculated the spray coefficients and Cartan connections for these metrics. They also identified the necessary and sufficient conditions for them to be projective and derived numerous new Douglas metrics from the existing ones. In 2025, N. Sadeghzadeh [115] introduced new classes of Finsler metrics, namely $\tilde{D}$ -stretch metrics, isotropic $\tilde{D}$ -stretch metrics, and relatively isotropic $\tilde{D}$ -metrics, by exploring the Douglas curvature in Finsler geometry. The research investigated the interplay between relatively isotropic $\tilde{D}$ -stretch metrics and other key curvatures, including $\tilde{E}$ -curvature and S -curvature, building upon prior studies on the relationships between Douglas curvature and these curvatures. In 2020, N. Sadeghzadeh and T. Rajabi [116] presented (α, β, γ) -metrics, discovered criteria for local projectivity, and demonstrated the metric's Douglas type.

Z. Shen [125] explored volume comparison in Riemann–Finsler geometry in 1997 and discussed about S -curvature, a non–Riemannian quantity. He derived an explicit formula for S -curvature in homogeneous Finsler manifold with an (α, β) -metric. In 2009, the expression for S -curvature in the homogeneous Randers metric was derived by S. Deng [44]. M. K. Gupta et al. obtained the expression for S -curvature in a homogeneous Finsler space with a generalized Matsumoto metric and demonstrated that a homogeneous generalized Matsumoto space with isotropic S -curvature must have vanishing S -curvature [27]. They also derived the expression for the mean Berwald curvature using the formula of S -curvature.

Rizza [109] proposed the concept of complex Finsler space in 1963. It is an extension of the classical setting that requires the fundamental function to be homogeneous for all complex scalars. The concept of R -complex Finsler spaces is relatively new in Finsler geometry. Munteanu and Purcaru extended the notion of complex Finsler spaces to R -complex Finsler spaces by restricting the scalars to $\lambda \in R$ [156]. The outcome was a new class of Finsler spaces called R -complex Finsler spaces. In 2020,

S. M. Vanithalakshmi, S. K. Narasimhamurthy, and M. K. Roopa [155] determined the fundamental tensor fields g_{ij} and $g_{i\bar{j}}$, along with their inverses and determinant. They also investigated the properties of the non-Hermitian R -complex Finsler space with the (α, β) -metric. G. Shanker and R. Sharma [138] defined the notion of R -complex Finsler space with the infinite series (α, β) -metric $\frac{\beta^2}{\beta - \alpha}$. They derived the fundamental tensor as well as its inverse and analyzed some fundamental properties of these spaces.

1.6 Scope and Applications of Finsler Geometry

Finsler geometry was initially utilized in gravitational theories, and this led to adjustments in the observational outcomes anticipated by general relativity. The primary application of Finsler geometry is in the geometrical interpretation of electromagnetism and gravity. Finsler geometry is a mathematical structure that extends Riemannian geometry by incorporating a broader set of metrics known as Finsler metrics. It has several applications in a variety of fields, including mathematics, physics, biology, computer science, and engineering. One of the applications of Finsler geometry in mathematics is in Differential geometry. It offers an expanded framework for investigating the geometric characteristics of manifolds, enabling the examination of curvature, geodesics, and various geometric structures that go beyond the scope of Riemannian geometry. Finsler geometry has also been applied in areas such as geometric measure theory, complex analysis, and differential equations. In the domain of statistics, Finsler geometry has been employed in the context of information geometry, which examines the geometric framework of statistical models and probability distributions.

Finsler geometry employs collections of Minkowski norms rather than collections of inner products to characterize geometric properties. This is completely analogous to the connection between Banach spaces and Hilbert spaces. Over the last decade, the field has experienced continuous modernization. In the past few years, numerous aspects of Finsler geometry have seen rapid advancement. This encompasses Finsler spaces with constant

curvature, along with the utilization of Finsler techniques in industrial and medical fields. Theoretical subjects tentatively cover Cartan spaces, the classification of Berwald spaces, Finsler spaces with constant flag curvature, Finsler-Einstein metrics, Finslerian volumes and measures, geodesic flows, Kobayashi metrics, Lagrange spaces, metric geometry, projective invariants, and rigidity theorems.

Finsler geometry holds promise for various applications within computer science, especially in areas such as computer graphics, image processing, and pattern recognition. It can offer geometric representations for analyzing shapes, tracking motion, and developing algorithms in computer vision. Within the field of economics, Finsler geometry has been utilized to represent and examine economic systems and decision-making processes. It offers a mathematical framework for investigating the geometric properties of utility functions, production frontiers, and resource distribution.

Finsler geometry is utilized in various engineering disciplines, including mechanical engineering, civil engineering, and aerospace engineering. It is applicable for examining the mechanics of solid substances, the performance of structures, and the enhancement of engineering designs. Finsler geometry has found applications in engineering domains including control theory, robotics, and optimization. It offers a mathematical framework for examining the dynamics, regulation, and enhancement of intricate systems.

The applications of Finsler geometry in physics can be primarily categorized into two areas. First, it serves as a useful geometric framework for describing point particle mechanics and point particle approximations of field theories, such as ray theory in various media, while also providing a geometric perspective on fluid dynamics. Second, there are efforts to utilize Finsler geometry as the framework for spacetime that explains gravity. The applications of Finsler geometry as spacetime geometry give rise to a number of questions concerning the equations used to determine the geometry of spacetime, the existence of geometric objects appearing therein, the description of observers, and the coupling of matter fields. Finsler space times incorporate causality, precise definitions of observers and measurements, field theories, and gravitational dynamics to determine outcomes. The shape of spacetime is determined by its matter field compositions.

Finsler geometry is used in biomechanics to analyze biological forms, morphometrics, and imaging. It allows for the analysis of shape deformations and quantification of biological structures. It has also been used to analyze the shape and structure of macromolecules such as proteins and DNA in molecular biology. In evolutionary biology, Finsler geometry has been used to study evolutionary processes and phylogenetic trees.

CHAPTER 2

On a Special Finsler (α, β) -Metric

We begin by examining the theory of projective transition amid two categories of (α, β) -metrics. The objective is to identify the conditions that define projective transformations between Finsler spaces equipped with (α, β) -metrics, as specified below:

(i) Cubic (α, β) -metric $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ and Kropina metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$,

(ii) Square root (α, β) -metric $F = \sqrt{\alpha(\alpha + \beta)}$ and Kropina metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$.

2.1 Introduction

In 1961, Rapcsák [105] studied the necessary and sufficient conditions for projective change, while in 1994, S. Bácsó and M. Matsumoto [15] studied the projective change between Finsler spaces with the (α, β) -metric. In 2008, H. S. Park and Y. Lee [100] investigated the projective change between a Finsler space with an (α, β) -metric and the corresponding Riemannian metric. Z. Shen and Civi Yildirim [125] investigated projectively flat metrics with constant flag curvature also in 2008. In 2009, N. Cui and Y. B. Shen [42] studied the projective change between two forms of (α, β) -metrics. N. Cui [41] investigated the S -curvature of several (α, β) -metrics in 2006. In 2009, Z. Lin [76] investigated (α, β) -metrics with constant flag curvature. The projective change between two Finsler spaces has been studied by several authors ([15, 42, 81, 96, 125]).

In 1929, L. Berwald constructed an example of a projectively flat Finsler metric of constant flag curvature $K = 0$, on the unit ball B^n , given by

$$F(x,y) = \frac{(\sqrt{(1-|x|^2)|y|^2 + \langle x,y \rangle^2} + \langle x,y \rangle)^2}{(1-|x|^2)^2 \sqrt{(1-|x|^2)|y|^2 + \langle x,y \rangle^2}}, (x,y) \in TR^n, \text{ where } x = x^i, y = y^i.$$

Here, $|\cdot|$ and $\langle \cdot, \cdot \rangle$ denote the standard Euclidean norm and inner product, respectively, on R^n .

Furthermore, L. Berwald introduced the Kropina metric for a two-dimensional Finsler space with a rectilinear extremal, which was later investigated by V. K. Kropina [66]. Kropina metrics are not regular Finsler metrics. Being the simplest non-trivial Finsler metrics, Kropina metrics have several intriguing applications in physics, electron optics with a magnetic field, dissipative mechanics, and irreversible thermodynamics [123]. They also have promising applications in relativistic field theory, control theory, evolutionary biology, and developmental biology.

2.1.1 Preliminaries

In this section, we cover some fundamental preliminary concepts for this chapter.

Two Finsler metrics F and $\tilde{F}$ are projectively related if and only if their spray coefficients satisfy the relation

$$G^i = \tilde{G}^i + P(x,y)y^i, \quad (2.1.1)$$

where G^i and $\tilde{G}^i$ are the spray coefficients of F and $\tilde{F}$ respectively and $P(x,y)$ is a scalar function on $TM \setminus \{0\}$, homogeneous of degree one in y .

In 1972, M. Matsumoto [79] introduced (α, β) -metrics. By definition, an (α, β) -metric is a Finsler metric of the following form:

$$F = \alpha \phi(s), \quad s = \frac{\beta}{\alpha} \quad [118].$$

Where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form satisfying $\|\beta_x\| < b_0, \forall x \in M$. It is generally known that $F = \alpha\phi(s)$ is a regular (α, β) -metric if the function $\phi(s)$ is a positive C^∞ function for $|s| < b_0$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \leq b < b_0. \quad (2.1.2)$$

In this case, F induces a positive definite metric tensor.

The relation between the spray coefficients G^i of F and the geodesic coefficients G_α^i of α is given by [122]

$$G^i = G_\alpha^i + \alpha Q s_0^i + \{-2Q\alpha s_0 + r_{00}\} \{\Psi b^i + \Theta \alpha^{-1} y^i\}, \quad (2.1.3)$$

where,

$$\begin{cases} \Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')}, \\ Q = \frac{\phi'}{\phi - s\phi'}, \\ \Psi = \frac{1}{2} \frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}. \end{cases}$$

The Douglas tensor is referred to as $D := D_{jkl}^i \partial_i \otimes dx^j \otimes dx^k \otimes dx^l$, where,

$$D_{jkl}^i := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left(G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i \right). \quad (2.1.4)$$

A Finsler metric is called a Douglas metric when the Douglas tensor vanishes ([42, 147]). The Douglas tensor is a projective invariant [96]. Since the spray coefficients of a Riemannian metric are quadratic forms, the Douglas tensor vanishes for the Riemannian metric, indicating that it is a non-Riemannian quantity. It is a fundamental fact that every Berwald metric is a Douglas metric.

To represent the appropriate quantities of the metric $\tilde{F}$, we use quantities with a tilde. We now compute the Douglas tensor of an (α, β) -metric.

Let

$$\tilde{G}^i = G_\alpha^i + \alpha Q s_0^i + \Psi \{-2Q\alpha s_0 + r_{00}\} b^i. \quad (2.1.5)$$

Hence, equation (2.1.3) can be written as

$$G^i = \tilde{G}^i + \Theta \{-2Q\alpha s_0 + r_{00}\} \alpha^{-1} y^i.$$

Clearly, if the sprays G^i and $\tilde{G}^i$ are projective equivalent, they will have the same Douglas tensor. Consider

$$T^i = \alpha Q s_0^i + \Psi \{-2Q\alpha s_0 + r_{00}\} b^i. \quad (2.1.6)$$

Differentiating equation (2.1.6) with respect to y^m , we get

$$T_{y^m}^m = Q' s_0 + \Psi' \alpha^{-1} (b^2 - s^2) [r_{00} - 2Q\alpha s_0] + 2\Psi [r_0 - Q' (b^2 - s^2) s_0 - Q s s_0]. \quad (2.1.7)$$

Equation (2.1.5) can be rewritten as $\tilde{G}^i = G_\alpha^i + T^i$. By equation (2.1.4), we get [118]

$$\begin{cases} \tilde{D}_{jkl}^i = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} (G_\alpha^i - \frac{1}{n+1} \frac{\partial G_\alpha^m}{\partial y^m} y^i + T^i - \frac{1}{n+1} \frac{\partial T^m}{\partial y^m} y^i) \\ = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left(T^i - \frac{1}{n+1} \frac{\partial T^m}{\partial y^m} y^i \right). \end{cases} \quad (2.1.8)$$

The Douglas tensor of an (α, β) -metric is given by the following equation

$$\tilde{D}_{jkl}^i = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left(T^i - \frac{1}{n+1} \frac{\partial T^m}{\partial y^m} y^i \right). \quad (2.1.9)$$

Therefore, if F and $\tilde{F}$ possess the same Douglas tensor, that is, to say $D_{jkl}^i = \tilde{D}_{jkl}^i$, then by equations (2.1.4) and (2.1.9), we get

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left[T^i - \tilde{T}^i - \frac{1}{n+1} (T_{y^m}^m - \tilde{T}_{y^m}^m) y^i \right] = 0. \quad (2.1.10)$$

Then there exists a class of scalar functions $H_{jk}^i := H_{jk}^i(x)$, such that

$$T^i - \tilde{T}^i - \frac{1}{n+1}(T_{y^m}^m - \tilde{T}_{y^m}^m)y^i = H_{00}^i, \quad (2.1.11)$$

where $H_{00}^i := H_{jk}^i(x)y^jy^k$, T^i and $T_{y^m}^m$ are given by relations (2.1.6) and (2.1.7), respectively.

2.1.2 Projective Transition of the Two Metrics

In this instance, we consider the projectively interrelated two (α, β) -metrics, namely the cubic (α, β) -metric $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ and the Kropina metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$.

For the cubic (α, β) -metric, one can prove by (2.1.2) that F is a regular Finsler metric if and only if $\|\beta_x\| < 1$ for any $x \in M$. The geodesic coefficients are given by equation (2.1.3) with.

$$\begin{cases} Q := \frac{3}{1-2s}, \\ \Theta := \frac{3-12s}{2(1-s-8s^2+6b^2)}, \\ \Psi := \frac{3}{1-s-8s^2+6b^2}. \end{cases} \quad (2.1.12)$$

The Kropina-metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ is not a regular (α, β) -metric, but the relation $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0$ remains valid for $|s| > 0$.

The geodesic coefficients are provided by equation (2.1.3)

$$\begin{cases} \tilde{Q} := \frac{-1}{2s}, \\ \tilde{\Theta} := \frac{-s}{b^2}, \\ \tilde{\Psi} := \frac{1}{2b^2}. \end{cases} \quad (2.1.13)$$

To simplify this research work, we assume that $\lambda := \frac{1}{(n+1)}$.

Lemma 2.1.1. [80] Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on M^n . Then

1. ($n \geq 3$) Kropina metric $F = \frac{\alpha^2}{\beta}$ with $b^2 \neq 0$ is a Douglas metric if and only if

$$s_{ik} = \frac{1}{b^2}(b_i s_k - b_k s_i). \quad (2.1.14)$$

2. ($n = 2$) Kropina metric F is a Douglas metric.

Furthermore, the Douglas tensor is projective invariant; hence, we have

Proposition 2.1.2. Let $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ be a cubic (α, β) -metric and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be a Kropina metric on a manifold M with dimension $n \geq 3$, where α and $\tilde{\alpha}$ are two Riemannian metrics, β and $\tilde{\beta}$ are two non-zero 1-forms. The Finsler metrics F and $\tilde{F}$ have the same Douglas tensor if and only if both are Douglas metrics.

Proof. The sufficiency is obvious; we simply need to prove the necessity. Suppose that F and $\tilde{F}$ have the same Douglas tensor on an n -dimensional manifold M when $n \geq 3$. Then $D^i_{jkl} = \tilde{D}^i_{jkl}$, which implies that (2.1.11) holds.

Substituting equations (2.1.12) and (2.1.13) into equation (2.1.11), we obtain

$$\left\{ \begin{array}{l} \frac{\alpha^{11}\xi_1 + \alpha^{10}\xi_2 + \alpha^9\xi_3 + \alpha^8\xi_4 + \alpha^7\xi_5 + \alpha^6\xi_6}{\alpha^{10}\eta_1 + \alpha^9\eta_2 + \alpha^8\eta_3 + \alpha^7\eta_4 + \alpha^6\eta_5 + \alpha^5\eta_6 + \alpha^4\eta_7 + \alpha^3\eta_8 + \alpha^2\eta_9 + \alpha\eta_{10} + \eta_{11}} \\ + \frac{\alpha^5\xi_7 + \alpha^4\xi_8 + \alpha^3\xi_9 + \alpha^2\xi_{10} + \alpha\xi_{11} + \xi_{12}}{\alpha^{10}\eta_1 + \alpha^9\eta_2 + \alpha^8\eta_3 + \alpha^7\eta_4 + \alpha^6\eta_5 + \alpha^5\eta_6 + \alpha^4\eta_7 + \alpha^3\eta_8 + \alpha^2\eta_9 + \alpha\eta_{10} + \eta_{11}} \\ + \frac{1}{2\tilde{b}^2\tilde{\beta}} [\tilde{\alpha}^2\tilde{\xi}_1 + \tilde{\xi}_2] = H_{00}^i \end{array} \right. \quad (2.1.15)$$

where,

$$\left\{ \begin{array}{l} \xi_1 = A_1 B_6, \\ \xi_2 = A_1 B_7 + A_2 B_6 - \lambda y^i A_5 B_1, \\ \xi_3 = A_1 B_8 + A_2 B_7 + A_3 B_6 - \lambda y^i A_5 B_2 - \lambda y^i A_6 B_1, \\ \xi_4 = A_1 B_9 + A_2 B_8 + A_3 B_7 + A_4 B_6 - \lambda y^i A_5 B_3 - \lambda y^i A_6 B_2 - \lambda y^i A_7 B_1, \\ \xi_5 = A_1 B_{10} + A_2 B_9 + A_3 B_8 + A_4 B_7 - \lambda y^i A_5 B_4 - \lambda y^i A_6 B_3 - \lambda y^i A_7 B_2 - \lambda y^i A_8 B_1, \end{array} \right.$$

$$\left\{ \begin{aligned}
& \xi_6 = A_1 B_{11} + A_2 B_{10} + A_3 B_9 + A_4 B_8 - \lambda y^i A_5 B_5 - \lambda y^i A_6 B_4 - \lambda y^i A_7 B_3 - \lambda y^i A_8 B_2 - y^i A_9 B_1, \\
& \xi_7 = A_1 B_{12} + A_2 B_{11} + A_3 B_{10} + A_4 B_9 - \lambda y^i A_6 B_5 - \lambda y^i A_7 B_4 - \lambda y^i A_8 B_3 - \lambda y^i A_9 B_2 - \lambda y^i A_{10} B_1, \\
& \xi_8 = A_2 B_{12} + A_3 B_{11} + A_4 B_{10} - \lambda y^i A_7 B_5 - \lambda y^i A_8 B_4 - \lambda y^i A_9 B_3 - \lambda y^i A_{10} B_2 - \lambda y^i A_{11} B_1, \\
& \xi_9 = A_3 B_{12} + A_4 B_{11} - \lambda y^i A_8 B_5 - \lambda y^i A_9 B_4 - \lambda y^i A_{10} B_3 - \lambda y^i A_{11} B_2, \\
& \xi_{10} = A_4 B_{12} - \lambda y^i A_9 B_5 - \lambda y^i A_{10} B_4 - \lambda y^i A_{11} B_3, \\
& \xi_{11} = -\lambda y^i A_{10} B_5 - \lambda y^i A_{11} B_4, \\
& \xi_{12} = -\lambda y^i A_{11} B_5, \\
& \eta_1 = B_1 B_6, \\
& \eta_2 = B_1 B_7 + B_2 B_6, \\
& \eta_3 = B_1 B_8 + B_2 B_7 + B_3 B_6, \\
& \eta_4 = B_1 B_9 + B_2 B_8 + B_3 B_7 + B_4 B_6, \\
& \eta_5 = B_1 B_{10} + B_2 B_9 + B_3 B_8 + B_4 B_7 + B_5 B_6, \\
& \eta_6 = B_1 B_{11} + B_2 B_{10} + B_3 B_9 + B_4 B_8 + B_5 B_7, \\
& \eta_7 = B_1 B_{12} + B_2 B_{11} + B_3 B_{10} + B_4 B_9 + B_5 B_8, \\
& \eta_8 = B_2 B_{12} + B_3 B_{11} + B_4 B_{10} + B_5 B_9, \\
& \eta_9 = B_3 B_{12} + B_4 B_{11} + B_5 B_{10}, \\
& \eta_{10} = B_4 B_{12} + B_5 B_{11}, \\
& \eta_{11} = B_5 B_{12}, \\
& \bar{\xi}_1 = \tilde{b}^2 \tilde{s}_0^i - \tilde{s}_0 \tilde{b}^i, \\
& \bar{\xi}_2 = 2\lambda y^i \tilde{\beta} \tilde{r}_{00} - \tilde{\beta} \tilde{r}_{00} \tilde{b}^i, \\
& A_1 = 3s_0^i + 18b^2 s_0^i - 18s_0 b^i, \\
& A_2 = -9\beta s_0^i - 36b^2 \beta s_0^i + 3r_{00} b^i + 36\beta s_0 b^i, \\
& A_3 = -18s_0^i \beta^2 - 12\beta r_{00} b^i, \\
& A_4 = 48\beta^3 s_0^i + 12\beta^2 r_{00} b^i, \\
& A_5 = s_0(6 + 18b^2) + r_0(6 + 36b^2), \\
& A_6 = s_0(-30\beta - 396\beta b^2) + 3b^2 r_{00} + r_0(-30\beta - 144b^2 \beta), \\
& A_7 = (18\beta^2 + 720b^2 \beta^2) s_0 + 36b^2 \beta r_{00} + 144b^2 \beta^2 r_0,
\end{aligned} \right.$$

$$\left\{ \begin{array}{l} A_8 = 420s_0\beta^3 + r_{00}(-3\beta^2 - 180b^2\beta^2) + r_0(168\beta^3), \\ A_9 = -768s_0\beta^4 + r_{00}(-36\beta^3 + 192b^2\beta^3) + r_0(-192\beta^4), \\ A_{10} = 180r_{00}\beta^4, \\ A_{11} = -192\beta^5, \\ B_1 = 1 + 6b^2, \\ B_2 = -5\beta - 24b^2\beta, \\ B_3 = 24b^2\beta^2, \\ B_4 = 28\beta^3, \\ B_5 = -32\beta^4, \\ B_6 = 1 + 12b^2 + 36b^4, \\ B_7 = -6\beta - 60b^2\beta - 144b^4\beta, \\ B_8 = -3\beta^2 + 144b^4\beta^2, \\ B_9 = 68\beta^3 + 336b^2\beta^3, \\ B_{10} = -60\beta^4 - 384b^2\beta^4, \\ B_{11} = -192\beta^5, \\ B_{12} = 256\beta^6. \end{array} \right.$$

Here $\lambda := \frac{1}{n+1}$. Furthermore, (2.1.15) is equal to

$$\left\{ \begin{array}{l} (\alpha^{11}\xi_1 + \alpha^{10}\xi_2 + \alpha^9\xi_3 + \alpha^8\xi_4 + \alpha^7\xi_5 + \alpha^6\xi_6 \\ + \alpha^5\xi_7 + \alpha^4\xi_8 + \alpha^3\xi_9 + \alpha^2\xi_{10} + \alpha\xi_{11} + \xi_{12}) \\ \times (2\tilde{b}^2\tilde{\beta}) + (\alpha^{10}\eta_1 + \alpha^9\eta_2 + \alpha^8\eta_3 + \alpha^7\eta_4 + \alpha^6\eta_5 \\ + \alpha^5\eta_6 + \alpha^4\eta_7 + \alpha^3\eta_8 + \alpha^2\eta_9 + \alpha\eta_{10} + \eta_{11}) \times \\ (\tilde{\alpha}^2\tilde{\xi}_1 + \tilde{\xi}_2) = H_{00}^i(2\tilde{b}^2\tilde{\beta}) \times (\alpha^{10}\eta_1 + \alpha^9\eta_2 + \alpha^8\eta_3 \\ + \alpha^7\eta_4 + \alpha^6\eta_5 + \alpha^5\eta_6 + \alpha^4\eta_7 + \alpha^3\eta_8 + \alpha^2\eta_9 + \alpha\eta_{10} + \eta_{11}). \end{array} \right. \quad (2.1.16)$$

By replacing y^i with $-y^i$ in (2.1.16), we get

$$\left\{ \begin{array}{l} (-\alpha^{11}\xi_1 + \alpha^{10}\xi_2 - \alpha^9\xi_3 + \alpha^8\xi_4 - \alpha^7\xi_5 + \alpha^6\xi_6 \\ -\alpha^5\xi_7 + \alpha^4\xi_8 - \alpha^3\xi_9 + \alpha^2\xi_{10} - \alpha\xi_{11} + \xi_{12}) \\ \times (-2\tilde{b}^2\tilde{\beta}) - (\alpha^{10}\eta_1 - \alpha^9\eta_2 + \alpha^8\eta_3 - \alpha^7\eta_4 + \alpha^6\eta_5 \\ -\alpha^5\eta_6 + \alpha^4\eta_7 - \alpha^3\eta_8 + \alpha^2\eta_9 - \alpha\eta_{10} + \eta_{11}) \times \\ (\tilde{\alpha}^2\bar{\xi}_1 + \bar{\xi}_2) = H_{00}^i(-2\tilde{b}^2\tilde{\beta}) \times (\alpha^{10}\eta_1 - \alpha^9\eta_2 + \alpha^8\eta_3 \\ -\alpha^7\eta_4 + \alpha^6\eta_5 - \alpha^5\eta_6 + \alpha^4\eta_7 - \alpha^3\eta_8 + \alpha^2\eta_9 - \alpha\eta_{10} + \eta_{11}). \end{array} \right. \quad (2.1.17)$$

Adding (2.1.16) with (2.1.17), we obtain .

$$\left\{ \begin{array}{l} (\alpha^{11}\xi_1 + \alpha^9\xi_3 + \alpha^7\xi_5 + \alpha^5\xi_7 + \alpha^3\xi_9 + \alpha\xi_{11}) \times (2\tilde{b}^2\tilde{\beta}) \\ + (\alpha^9\eta_2 + \alpha^7\eta_4 + \alpha^5\eta_6 + \alpha^3\eta_8 + \alpha\eta_{10}) \times (\tilde{\alpha}^2\bar{\xi}_1 + \bar{\xi}_2) \\ = H_{00}^i(2\tilde{b}^2\tilde{\beta}) \times (\alpha^9\eta_2 + \alpha^7\eta_4 + \alpha^5\eta_6 + \alpha^3\eta_8 + \alpha\eta_{10}). \end{array} \right. \quad (2.1.18)$$

Subtracting (2.1.17) from (2.1.16), we get.

$$\left\{ \begin{array}{l} (\alpha^{10}\xi_2 + \alpha^8\xi_4 + \alpha^6\xi_6 + \alpha^4\xi_8 + \alpha^2\xi_{10} + \xi_{12}) \times (2\tilde{b}^2\tilde{\beta}) + \\ (\alpha^{10}\eta_1 + \alpha^8\eta_3 + \alpha^6\eta_5 + \alpha^4\eta_7 + \alpha^2\eta_9 + \eta_{11}) \times (\tilde{\alpha}^2\bar{\xi}_1 + \bar{\xi}_2) \\ = H_{00}^i(2\tilde{b}^2\tilde{\beta}) \times (\alpha^{10}\eta_1 + \alpha^8\eta_3 + \alpha^6\eta_5 + \alpha^4\eta_7 + \alpha^2\eta_9 + \eta_{11}). \end{array} \right. \quad (2.1.19)$$

From (2.1.18) we have,

$$\left\{ \begin{array}{l} (\alpha^{10}\xi_1 + \alpha^8\xi_3 + \alpha^6\xi_5 + \alpha^4\xi_7 + \alpha^2\xi_9 + \xi_{11}) \times (2\tilde{b}^2\tilde{\beta}) \\ + (\alpha^8\eta_2 + \alpha^6\eta_4 + \alpha^4\eta_6 + \alpha^2\eta_8 + \eta_{10}) \times (\tilde{\alpha}^2\bar{\xi}_1 + \bar{\xi}_2) \\ = H_{00}^i(2\tilde{b}^2\tilde{\beta}) \times (\alpha^8\eta_2 + \alpha^6\eta_4 + \alpha^4\eta_6 + \alpha^2\eta_8 + \eta_{10}). \end{array} \right. \quad (2.1.20)$$

According to equation (2.1.20), the term $(\alpha^8\eta_2 + \alpha^6\eta_4 + \alpha^4\eta_6 + \alpha^2\eta_8 + \eta_{10}) \times (\tilde{\alpha}^2\bar{\xi}_1 + \bar{\xi}_2)$ can be divided by $\tilde{\beta}$. The expression $\tilde{\alpha}^2\bar{\xi}_1\alpha^8\eta_2$ is divisible by $\tilde{\beta}$. Since $\beta = \mu\tilde{\beta}$ and $\tilde{\beta}$ is prime with respect to α and $\tilde{\alpha}$; thus, $\bar{\xi}_1 := \tilde{b}^2\tilde{s}_0^i - \tilde{b}^i\tilde{s}_0$ is divisible by $\tilde{\beta}$. Consequently, there exists a scalar function $\psi^i(x)$ such that

$$\tilde{b}^2 \tilde{s}_0^i - \tilde{b}^i \tilde{s}_0 = \tilde{\beta} \psi^i. \quad (2.1.21)$$

Contracting (2.1.21) by $\tilde{y}_i := \tilde{a}_{ij} y^j$, we obtain $\psi^i(\tilde{y}_i) = -\tilde{s}_0$. As $\tilde{y}_i$ is an arbitrary vector, we get $\psi^i(x) = -\tilde{s}^i$. Then We have,

$$\tilde{s}_{ij} = \frac{1}{\tilde{b}^2} [\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i], \quad (2.1.22)$$

given that $\tilde{b}^2 \neq 0$. Lemma (2.1.1) states that the Douglas metric is $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$. F and $\tilde{F}$ are both Douglas metrics because they have the same Douglas tensor.

When $n = 2$, $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ is a Douglas metric according to Lemma (2.1.1). Since F and $\tilde{F}$ have the same Douglas tensor, they are all Douglas metrics. This concludes the demonstration of proposition (2.1.2).

Now we are able to claim the following theorem.

Theorem 2.1.3. *Let $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be two (α, β) -metrics on a manifold M with dimension $n > 2$, where α and $\tilde{\alpha}$ are two Riemannian metrics, $\beta = b_i y^i$ and $\tilde{\beta}$ are two non-zero colinear 1-forms. Then F is projectively related to $\tilde{F}$ if and only if the following relations hold:*

$$\begin{aligned} (i) \quad G_\alpha^i &= \tilde{G}_\alpha^i + \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2 \tilde{s}_i + r_{00} \tilde{b}^i] + \theta y^i - \frac{\tau \alpha^2 ((1 + 6b^2)\alpha^2 - 8\beta^2) b^i}{\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2 \alpha^2}, \\ (ii) \quad b_{i|j} &= \frac{\tau}{3} \{ (1 + 6b^2) a_{ij} - 8b_i b_j \}, \\ (iii) \quad \tilde{s}_{ij} &= \frac{1}{\tilde{b}^2} [\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i], \end{aligned}$$

where $\tau = \tau(x)$ is a scalar function, $b_{i|j}$ denotes the coefficients of the covariant derivative of β with respect to α and $\tilde{b}^i := \tilde{a}^{ij} \tilde{b}_j$, $\tilde{b} := \|\tilde{\beta}\|_{\tilde{\alpha}}$, and $\theta = \theta_i y^i$ is a 1-form on M .

The cubic (α, β) -metric $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ is a Douglas metric if and only if $b_{i|j} = \frac{\tau}{3} \{ (1 + 6b^2) a_{ij} - 8b_i b_j \}$ holds for a scalar $\tau = \tau(x)$. It is known that the Kropina metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ is a Douglas metric if and only if $\tilde{s}_{ij} = \frac{1}{\tilde{b}^2} [\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i]$.

Proof. First, we establish the necessity. The Douglas tensor is defined to be invariant under projective change between two Finsler metrics. If F is projectively related to $\tilde{F}$, then they have the same Douglas tensor. By Proposition (2.1.2), we obtain that F and $\tilde{F}$ are both Douglas metrics. It has been proved in [80] that the Kropina metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ is a Douglas metric if and only if

$$\tilde{s}_{ij} = \frac{1}{\tilde{b}^2} [\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i]. \quad (2.1.23)$$

In the present chapter, we prove that cubic (α, β) -metric $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ is a Douglas metric if and only if

$$b_{i|j} = \frac{\tau}{3} \{ (1 + 6b^2)a_{ij} - 8b_i b_j \}. \quad (2.1.24)$$

For some scalar function $\tau = \tau(x)$, where $b_{i|j}$ denotes the coefficients of the covariant derivative of β with respect to α . Furthermore, it follows that β is a closed 1-form. Substituting (2.1.24) and (2.1.12) into (2.1.3), we have

$$G^i = G_\alpha^i + \frac{\tau \alpha^2 ((1 + 6b^2)\alpha^2 - 8\beta^2) b^i}{\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2 \alpha^2} + \frac{\tau ((1 + 6b^2)\alpha^2 - 8\beta^2) (\alpha - 4\beta) y^i}{2(\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2 \alpha^2)}. \quad (2.1.25)$$

On the other hand, Putting (2.1.23) and (2.1.13) into (2.1.3), we have

$$\tilde{G}^i = \tilde{G}_{\tilde{\alpha}}^i - \frac{1}{2\tilde{b}^2} \left[-\tilde{\alpha}^2 \tilde{s}^i + (2\tilde{s}_0 y^i - \tilde{r}_{00} \tilde{b}^i) + \frac{2\tilde{r}_{00} \tilde{\beta} y^i}{\tilde{\alpha}^2} \right]. \quad (2.1.26)$$

Since F is projectively related to $\tilde{F}$, by [118], there exists a scalar function $P = P(x, y)$ on $TM \setminus \{0\}$ such that

$$G^i = \tilde{G}^i + P y^i. \quad (2.1.27)$$

Combining (2.1.25), (2.1.26), and (2.1.27), we obtain

$$\left\{ \begin{aligned} & \left[P - \frac{\tau((1+6b^2)\alpha^2 - 8\beta^2)(\alpha - 4\beta)}{2(\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2\alpha^2)} - \frac{1}{\tilde{b}^2} \left(\tilde{s}_0 + \frac{\tilde{r}_{00}\tilde{\beta}}{\tilde{\alpha}^2} \right) \right] y^i \\ & = G_\alpha^i + \frac{\tau\alpha^2((1+6b^2)\alpha^2 - 8\beta^2)b^i}{\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2\alpha^2} - \tilde{G}_\alpha^i - \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2\tilde{s}^i + \tilde{r}_{00}\tilde{b}^i]. \end{aligned} \right. \quad (2.1.28)$$

The right-hand side of the above equation is a quadratic form in y . Consequently, there exists a unique 1-form $\theta = \theta_i(x)y^i$ on M such that

$$P - \frac{\tau((1+6b^2)\alpha^2 - 8\beta^2)(\alpha - 4\beta)}{2(\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2\alpha^2)} - \frac{1}{\tilde{b}^2} \left(\tilde{s}_0 + \frac{\tilde{r}_{00}\tilde{\beta}}{\tilde{\alpha}^2} \right) = \theta. \quad (2.1.29)$$

Then, we have

$$G_\alpha^i = \tilde{G}_\alpha^i - \frac{\tau\alpha^2((1+6b^2)\alpha^2 - 8\beta^2)b^i}{\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2\alpha^2} + \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2\tilde{s}^i + \tilde{r}_{00}\tilde{b}^i] + \theta y^i. \quad (2.1.30)$$

This, together with (2.1.23), (2.1.24), and (2.1.30), completes the proof of the necessity. Conversely, plugging (2.1.24) into (2.1.3) with (2.1.12) yields (2.1.25). Plugging (2.1.23) into (2.1.3) with (2.1.13) yields (2.1.26). From (2.1.25) and (2.1.30) we have,

$$G^i = \tilde{G}_\alpha^i + \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2\tilde{s}^i + \tilde{r}_{00}\tilde{b}^i] + y^i \left[\theta + \frac{\tau((1+6b^2)\alpha^2 - 8\beta^2)(\alpha - 4\beta)}{2(\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2\alpha^2)} \right]. \quad (2.1.31)$$

Putting the value of $\tilde{G}_\alpha^i$ from equation (2.1.26) to equation (2.1.31), we get

$$G^i = \tilde{G}^i + \left\{ \left(\frac{1}{2\tilde{b}^2} \left[2\tilde{s}_0 + \frac{2\tilde{r}_{00}\tilde{\beta}}{\tilde{\alpha}^2} \right] \right) + \left(\theta + \frac{\tau((1+6b^2)\alpha^2 - 8\beta^2)(\alpha - 4\beta)}{2(\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2\alpha^2)} \right) \right\} \times y^i. \quad (2.1.32)$$

From equation (2.1.27), we get

$$P = \theta + \frac{\tau((1+6b^2)\alpha^2 - 8\beta^2)(\alpha - 4\beta)}{2(\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2\alpha^2)} + \frac{1}{2\tilde{b}^2} \left[2\tilde{s}_0 + \frac{2\tilde{r}_{00}\tilde{\beta}}{\tilde{\alpha}^2} \right]. \quad (2.1.33)$$

that is, F is projectively related to $\tilde{F}$, thereby completing the proof of Theorem (2.1.3). $\square$

□

Corollary 2.1.4. *Let $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be (α, β) -metrics defined on an n -dimensional manifold M with $n > 2$, where α and $\tilde{\alpha}$ are Riemannian metrics, and $\beta, \tilde{\beta}$ are non-zero 1-forms. Then F is projectively related to $\tilde{F}$ if and only if both metrics are of Douglas type and the spray coefficients of α and $\tilde{\alpha}$ satisfy the following relation*

$$G_{\alpha}^i = \tilde{G}_{\tilde{\alpha}}^i + \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2 \tilde{s}_i + r_{00} \tilde{b}^i] + \theta y^i - \frac{\tau \alpha^2 ((1 + 6b^2)\alpha^2 - 8\beta^2) b^i}{\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2\alpha^2}, \quad (2.1.34)$$

where $\tilde{b}^i := \tilde{\alpha}^{ij} \tilde{b}_j$, $\tau = \tau(x)$ is a scalar function and $\theta = \theta_i y^i$ is a 1-form on M .

2.1.3 Projective Change with Curvature Properties

For a Finsler metric F , the Berwald curvature tensor is defined as:

$$\mathbf{B} := B_{jkl}^i dx^j \otimes \partial_i \otimes dx^k \otimes dx^l,$$

where $B_{jkl}^i = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} = [G^i]_{y^j y^k y^l}$.

A Finsler metric F is said to possess isotropic Berwald curvature if

$$B_{jkl}^i = c(F_{y^j y^k} \delta_l^i + F_{y^j y^l} \delta_k^i + F_{y^k y^l} \delta_j^i + F_{y^j y^k y^l} y^i),$$

where $c = c(x)$ is a scalar function on M .

The mean Berwald curvature tensor is defined by $E := E_{ij} dx^i \otimes dx^j$, where $E_{ij} := \frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j}$

$$\left(\frac{\partial G^m}{\partial y^m} \right) = \frac{1}{2} B_{mij}^m.$$

A Finsler metric is said to be isotropic mean Berwald curvature if $E_{ij} = \frac{n+1}{2} c(x) F_{y^i y^j}$,

where $c = c(x)$ is a scalar function on M [124]. Clearly, the Finsler metric of isotropic Berwald curvature must be of isotropic mean Berwald curvature.

A Finsler metric F is said to have isotropic S -curvature if $\mathbf{S} = (n+1)c(x)F$ for some scalar

function $c(x)$ on M .

Furthermore, N.W. Cui proves the following:

Lemma 2.1.5. [41] *For metric $F = \alpha + \varepsilon\beta + k\frac{\beta^2}{\alpha} + r\frac{\beta^3}{\alpha^2}$, where $\varepsilon, k, r \neq 0$ are constants on an n -dimensional manifold M , the following are equivalent:*

- (i) F is of isotropic S -curvature, that is, $S = (n+1)c(x)F$;
- (ii) F is of isotropic mean Berwald curvature, $E = \frac{n+1}{2}c(x)F^{-1}h$;
- (iii) F has vanished S -curvature, that is, $S = 0$;
- (iv) F is a weakly Berwald metric, that is, $E = 0$;
- (v) β is a Killing 1-form of constant length with respect to α , that is, $r_{00} = s_0 = 0$, where $c = c(x)$ is a scalar function.

The Lemma (2.1.5) is valid for $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ when we put $\varepsilon = 3, k = 3$, and $r = 1$.

In the present section, we suppose that $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ has some curvature properties. The Kropina metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ is projectively related to F . For $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ of isotropic S -curvature, we have the following:

Theorem 2.1.6. *Let $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be two (α, β) -metrics on an n -dimensional manifold M , where $n > 2$, α and $\tilde{\alpha}$ are Riemannian metrics, and $\beta = b_i y^i$ and $\tilde{\beta}$ are non-zero 1-forms. Assume that F admits isotropic S -curvature. Then F is projectively related to $\tilde{F}$ if and only if the following conditions are satisfied:*

- (i) $G_\alpha^i = \tilde{G}_\alpha^i + \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2 \tilde{s}^i + \tilde{r}_{00} \tilde{b}^i] + \theta y^i$;
- (ii) β is parallel with respect to α , that is, $b_{i|j} = 0$;
- (iii) $\tilde{s}_{ij} = \frac{1}{\tilde{b}^2} [\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i]$, where $b_{i|j}$ denotes the coefficients of the covariant derivatives of β with respect to α and $\tilde{b}^i := \tilde{a}^{ij} \tilde{b}_j$, $\tilde{b} := \|\tilde{\beta}\|_{\tilde{\alpha}}$ and $\theta = \theta_i y^i$ is a 1-form on M .

Proof. The sufficiency is an immediate consequence of Theorem (2.1.3), which asserts that if $\tilde{F}$ is projectively related to F , then

$$b_{i|j} = \frac{\tau}{3} [(1 + 6b^2)a_{ij} - 8b_i b_j],$$

for a scalar function $\tau = \tau(x)$. Upon contracting the preceding equation with y^i and y^j , one obtains

$$r_{00} = \frac{\tau}{3} [(1 + 6b^2)\alpha^2 - 8\beta^2]. \quad (2.1.35)$$

According to Lemma (2.1.5), if F possesses isotropic S -curvature, then $r_{00} = s_0 = 0$.

Plugging $r_{00} = s_0 = 0$ in equation (2.1.35), we get

$$(1 + 6b^2)\alpha^2 - 8\beta^2 = 0,$$

provided $\tau \neq 0$. That is

$$(1 + 6b^2)a_{ij} - 8b_i b_j = 0.$$

Upon contracting the preceding equation with a^{ij} , one obtains $n + (6n - 8)b^2 = 0$, which yields a contradiction. Hence, we must have $\tau = 0$, which cannot hold for any $n \geq 2$. An application of Theorem (2.1.3) therefore completes the proof.

We know that the Finsler metric for isotropic Berwald curvature necessarily possesses isotropic mean Berwald curvature. As a result of Lemma (2.1.5), and assuming that F has isotropic Berwald curvature, the theorem is consequently obtained. $\square$

Theorem 2.1.7. *Let $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be (α, β) -metrics defined on an n -dimensional manifold M , where $n > 2$, α and $\tilde{\alpha}$ are Riemannian metrics, and $\beta = b_i y^i$, $\tilde{\beta}$ are non-zero 1-forms. Under the assumption that F admits isotropic Berwald curvature, F is projectively related to $\tilde{F}$ if and only if the following conditions are fulfilled:*

$$1. \quad G_{\alpha}^i = \tilde{G}_{\tilde{\alpha}}^i + \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2 \tilde{s}^i + \tilde{r}_{00} \tilde{b}^i] + \theta y^i;$$

2. β is parallel with respect to α , that is, $b_{i|j} = 0$;

$$3. \tilde{s}_{ij} = \frac{1}{\tilde{b}^2} [\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i].$$

Here, $b_{i|j}$ denotes the coefficients of the covariant derivative of β with respect to α and $\tilde{b}^i := \tilde{a}^{ij} \tilde{b}_j$, $\tilde{b} := ||\tilde{\beta}||_{\tilde{\alpha}}$ and $\theta = \theta_i y^i$ is a 1-form on M .

Theorem 2.1.8. Let $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ be projectively related to $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ and $\tilde{F}$ has isotropic Berwald curvature. Then F has isotropic Berwald curvature if and only if F has isotropic S -curvature.

Proof. Suppose F has isotropic Berwald curvature, then F has isotropic mean Berwald curvature. Thus, by Lemma (2.1.5), F is of isotropic S -curvature. This establishes the sufficient condition. Since F and $\tilde{F}$ are projectively related, (2.1.1) holds. Assume that Finsler metric F possesses isotropic S -curvature $S = (n+1)c(x)F$. According to Lemma (2.1.5), F has isotropic mean Berwald curvature, which means that $E_{ij} = \frac{n+1}{2} cF_{y^i y^j}$.

Given that $\tilde{F}$ has isotropic Berwald curvature, then

$$\tilde{B}_{jkl}^i = \tilde{c} \left(\tilde{F}_{y^j y^k} \delta_l^i + \tilde{F}_{y^j y^l} \delta_k^i + \tilde{F}_{y^k y^l} \delta_j^i + \tilde{F}_{y^j y^k y^l} y^i \right),$$

where $\tilde{c} = \tilde{c}(x)$ is a scalar function on M . By virtue of the definition of the mean Berwald tensor, it follows from (2.1.1) that $cF_{y^i y^j} = \tilde{c}\tilde{F}_{y^i y^j} + P_{y^i y^j}$, from which, in view of [118], one obtains

$$cF_{y^i y^j y^k} = \tilde{c}\tilde{F}_{y^i y^j y^k} + P_{y^i y^j y^k}.$$

Consequently, we obtain

$$\left\{ \begin{aligned} B_{jkl}^i &= \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} \\ &= \tilde{B}_{jkl}^i + (P_{y^j y^k} \delta_l^i + P_{y^j y^l} \delta_k^i + P_{y^k y^l} \delta_j^i + P_{y^j y^k y^l} y^i) \\ &= \tilde{c}(\tilde{F}_{y^j y^k} \delta_l^i + \tilde{F}_{y^j y^l} \delta_k^i + \tilde{F}_{y^k y^l} \delta_j^i + \tilde{F}_{y^j y^k y^l} y^i) \\ &\quad + (P_{y^j y^k} \delta_l^i + P_{y^j y^l} \delta_k^i + P_{y^k y^l} \delta_j^i + P_{y^j y^k y^l} y^i) \\ &= c(F_{y^j y^k} \delta_l^i + F_{y^j y^l} \delta_k^i + F_{y^k y^l} \delta_j^i + F_{y^j y^k y^l} y^i). \end{aligned} \right.$$

This implies that F has isotropic Berwald curvature. We complete the proof. By the above methods, we could obtain the theorem. $\square$

Theorem 2.1.9. *Let $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ be projectively related to $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ and F has isotropic Berwald curvature. Then $\tilde{F}$ has isotropic Berwald curvature if and only if $\tilde{F}$ has isotropic S -curvature.*

2.2 Projective Transformation Between Square root (α, β) -Metric and Kropina Metric

The present section is concerned with an investigation of two projectively related (α, β) -metrics, specifically the square root (α, β) -metric $F = \sqrt{\alpha(\alpha + \beta)}$ and the Kropina metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$.

The geodesic coefficients satisfy (2.1.3), where the scalar functions Q , Θ , and Ψ are defined as follows [71]:

$$\left\{ \begin{aligned} Q &:= \frac{1}{s+2}, \\ \Theta &:= \frac{1+s}{2(1+s)(s+2) + s^2 - b^2}, \\ \Psi &:= \frac{-1}{6s^2 + 12s + 8 - 2b^2}. \end{aligned} \right. \quad (2.2.1)$$

The Kropina-metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ is not a regular (α, β) -metric, but the relation $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0$ remains valid for $|s| > 0$.

The geodesic coefficients are expressed as follows:

$$\begin{cases} \tilde{Q} := \frac{-1}{2s}, \\ \tilde{\Theta} := \frac{-s}{\tilde{b}^2}, \\ \tilde{\Psi} := \frac{1}{2\tilde{b}^2}. \end{cases} \quad (2.2.2)$$

Throughout the remainder of this section, we adopt the notation $\lambda := \frac{1}{n+1}$ to simplify subsequent expressions.

Proposition 2.2.1. *Let $F = \sqrt{\alpha(\alpha + \beta)}$ be a square root (α, β) -metric and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be a Kropina metric on a manifold M with dimension $n \geq 3$, where α and $\tilde{\alpha}$ are two Riemannian metrics, β and $\tilde{\beta}$ are two non-zero 1-forms. The Finsler metrics F and $\tilde{F}$ have same Douglas tensor if and only if both are Douglas metrics.*

Proof. Although the sufficiency is immediate, it remains to establish the necessity. Suppose that F and $\tilde{F}$ share the same Douglas tensor on an n -dimensional manifold M . It then follows that $D_{jkl}^i = \tilde{D}_{jkl}^i$, which implies that (2.1.11) holds.

On substituting (2.2.1) and (2.2.2) into (2.1.11), the following result is obtained:

$$\begin{cases} \frac{\alpha^{11}\chi_1 + \alpha^{10}\chi_2 + \alpha^9\chi_3 + \alpha^8\chi_4 + \alpha^7\chi_5 + \alpha^6\chi_6}{\alpha^{10}\pi_1 + \alpha^9\pi_2 + \alpha^8\pi_3 + \alpha^7\pi_4 + \alpha^6\pi_5 + \alpha^5\pi_6 + \alpha^4\pi_7 + \alpha^3\pi_8 + \alpha^2\pi_9 + \alpha\pi_{10} + \pi_{11}} \\ + \frac{\alpha^5\chi_7 + \alpha^4\chi_8 + \alpha^3\chi_9 + \alpha^2\chi_{10} + \alpha\chi_{11} + \chi_{12}}{\alpha^{10}\pi_1 + \alpha^9\pi_2 + \alpha^8\pi_3 + \alpha^7\pi_4 + \alpha^6\pi_5 + \alpha^5\pi_6 + \alpha^4\pi_7 + \alpha^3\pi_8 + \alpha^2\pi_9 + \alpha\pi_{10} + \pi_{11}} \\ + \frac{1}{2\tilde{b}^2\tilde{\beta}} [\tilde{\alpha}^2\tilde{\chi}_1 + \tilde{\chi}_2] = H_{00}^i \end{cases} \quad (2.2.3)$$

where,

$$\left\{ \begin{array}{l}
\chi_1 = A_1 K'_1, \\
\chi_2 = A_1 K'_2 + A_2 K'_1 - \lambda y^i K_1 A'_1, \\
\chi_3 = A_1 K'_3 + A_3 K'_1 + A_2 K'_2 - \lambda y^i K_1 A'_2 - \lambda y^i K_2 A'_1, \\
\chi_4 = A_1 K'_4 + A_2 K'_3 + A_3 K'_2 + A_4 K'_1 - \lambda y^i K_1 A'_3 - \lambda y^i K_2 A'_2 - \lambda y^i K_3 A'_1, \\
\chi_5 = A_1 K'_5 + A_2 K'_4 + A_3 K'_3 + A_4 K'_2 - \lambda y^i K_1 A'_4 - \lambda y^i K_2 A'_3 - \lambda y^i K_3 A'_2 - \lambda y^i K_4 A'_1, \\
\chi_6 = A_1 K'_6 + A_2 K'_5 + A_3 K'_4 + A_4 K'_3 - \lambda y^i K_1 A'_5 - \lambda y^i K_2 A'_4 - \lambda y^i K_3 A'_3 - \lambda y^i K_4 A'_2 \\
- \lambda y^i K_5 A'_1, \\
\chi_7 = A_1 K'_7 + A_2 K'_6 + A_3 K'_5 + A_4 K'_4 - \lambda y^i K_2 A'_5 - \lambda y^i K_3 A'_4 - \lambda y^i K_4 A'_3 - \lambda y^i K_5 A'_2 \\
- \lambda y^i K_6 A'_1, \\
\chi_8 = A_2 K'_7 + A_3 K'_6 + A_4 K'_5 - \lambda y^i K_3 A'_5 - \lambda y^i K_4 A'_4 - \lambda y^i K_5 A'_3 - \lambda y^i K_6 A'_2 - \lambda y^i K_7 A'_1, \\
\chi_9 = A_3 K'_7 + A_4 K'_6 - \lambda y^i K_4 A'_5 - \lambda y^i K_5 A'_4 - \lambda y^i K_6 A'_3 - \lambda y^i K_7 A'_2, \\
\chi_{10} = A_4 K'_7 - \lambda y^i K_5 A'_5 - \lambda y^i K_6 A'_4 - \lambda y^i K_7 A'_3, \\
\chi_{11} = -\lambda y^i K_6 A'_5 - \lambda y^i K_7 A'_4, \\
\chi_{12} = -\lambda y^i K_7 A'_5, \\
\pi_1 = A'_1 K'_1, \\
\pi_2 = A'_1 K'_2 + A'_2 K'_1, \\
\pi_3 = A'_1 K'_3 + A'_2 K'_2 + A'_3 K'_1, \\
\pi_4 = A'_1 K'_4 + A'_2 K'_3 + A'_3 K'_2 + A'_4 K'_1, \\
\pi_5 = A'_1 K'_5 + A'_2 K'_4 + A'_3 K'_3 + A'_4 K'_2 + A'_5 K'_1, \\
\pi_6 = A'_1 K'_6 + A'_2 K'_5 + A'_3 K'_4 + A'_4 K'_3 + A'_5 K'_2, \\
\pi_7 = A'_1 K'_7 + A'_2 K'_6 + A'_3 K'_5 + A'_4 K'_4 + A'_5 K'_3, \\
\pi_8 = A'_2 K'_7 + A'_3 K'_6 + A'_4 K'_5 + A'_5 K'_4, \\
\pi_9 = A'_3 K'_7 + A'_4 K'_6 + A'_5 K'_5, \\
\pi_{10} = A'_4 K'_7 + A'_5 K'_6, \\
\pi_{11} = A'_5 K'_7,
\end{array} \right.$$

$$\left\{ \begin{array}{l}
\bar{\chi}_1 = \tilde{b}^2 \tilde{s}_0^i - \tilde{s}_0 \tilde{b}^i, \\
\bar{\chi}_2 = 2\lambda y^i \tilde{\beta} \tilde{r}_{00} - \tilde{\beta} \tilde{r}_{00} \tilde{b}^i, \\
A_1 = -4b^2 s_0^i + 4s_0 b^i + 16s_0^i, \\
A_2 = 32\beta s_0^i - 2b^2 \beta s_0^i - 4r_{00} b^i + 2\beta s_0 b^i, \\
A_3 = 24s_0^i \beta^2 - 4\beta r_{00} b^i, \\
A_4 = 6\beta^3 s_0^i - \beta^2 r_{00} b^i, \\
A_1' = -8b^2 + 32, \\
A_2' = 80\beta - 8b^2 \beta, \\
A_3' = -2b^2 \beta^2 + 80\beta^2, \\
A_4' = 36\beta^3, \\
A_5' = 6\beta^4, \\
K_1 = 64s_0 - 8b^4 s_0 - 64r_0 + 16b^2 r_0, \\
K_2 = -160\beta s_0 - 8b^2 \beta s_0 + 48b^2 r_{00} - 160\beta r_0 + 16b^2 \beta r_0, \\
K_3 = -112\beta^2 s_0 + 4b^2 \beta^2 s_0 + 96b^2 \beta r_{00} - 160\beta^2 r_0 + 4b^2 \beta^2 r_0, \\
K_4 = -10s_0 \beta^3 + 60b^2 \beta^2 r_{00} - 72\beta^3 r_0 - 48\beta^2 r_{00}, \\
K_5 = 12\beta^4 s_0 + 12b^2 \beta^3 r_{00} - 96\beta^3 r_{00} - 12\beta^4 r_0, \\
K_6 = -60\beta^4 r_{00}, \\
K_7 = -12\beta^5 r_{00}, \\
K_1' = 256 - 128b^2 + 16b^4, \\
K_2' = 1024\beta - 320b^2 \beta + 16b^4 \beta, \\
K_3' = 1792\beta^2 - 320b^2 \beta^2 + 4b^4 \beta^2, \\
K_4' = 1768\beta^3 - 144b^2 \beta^3, \\
K_5' = 1000\beta^4 - 24b^2 \beta^4, \\
K_6' = 298\beta^5, \\
K_7' = \beta^6.
\end{array} \right.$$

Here $\lambda := \frac{1}{n+1}$. In addition, the value of (2.2.3) is equivalent to

$$\left\{ \begin{array}{l} (\alpha^{11}\chi_1 + \alpha^{10}\chi_2 + \alpha^9\chi_3 + \alpha^8\chi_4 + \alpha^7\chi_5 + \alpha^6\chi_6 \\ + \alpha^5\chi_7 + \alpha^4\chi_8 + \alpha^3\chi_9 + \alpha^2\chi_{10} + \alpha\chi_{11} + \chi_{12}) \\ \times (2\tilde{b}^2\tilde{\beta}) + (\alpha^{10}\pi_1 + \alpha^9\pi_2 + \alpha^8\pi_3 + \alpha^7\pi_4 + \alpha^6\pi_5 \\ + \alpha^5\pi_6 + \alpha^4\pi_7 + \alpha^3\pi_8 + \alpha^2\pi_9 + \alpha\pi_{10} + \pi_{11}) \times \\ (\tilde{\alpha}^2\tilde{\chi}_1 + \tilde{\chi}_2) = H_{00}^i(2\tilde{b}^2\tilde{\beta}) \times (\alpha^{10}\pi_1 + \alpha^9\pi_2 + \alpha^8\pi_3 \\ + \alpha^7\pi_4 + \alpha^6\pi_5 + \alpha^5\pi_6 + \alpha^4\pi_7 + \alpha^3\pi_8 + \alpha^2\pi_9 + \alpha\pi_{10} + \pi_{11}). \end{array} \right. \quad (2.2.4)$$

Substituting y^i with $-y^i$ in equation (2.2.4) yields the following result:

$$\left\{ \begin{array}{l} (-\alpha^{11}\chi_1 + \alpha^{10}\chi_2 - \alpha^9\chi_3 + \alpha^8\chi_4 - \alpha^7\chi_5 + \alpha^6\chi_6 \\ - \alpha^5\chi_7 + \alpha^4\chi_8 - \alpha^3\chi_9 + \alpha^2\chi_{10} - \alpha\chi_{11} + \chi_{12}) \\ \times (-2\tilde{b}^2\tilde{\beta}) - (\alpha^{10}\pi_1 - \alpha^9\pi_2 + \alpha^8\pi_3 - \alpha^7\pi_4 + \alpha^6\pi_5 \\ - \alpha^5\pi_6 + \alpha^4\pi_7 - \alpha^3\pi_8 + \alpha^2\pi_9 - \alpha\pi_{10} + \pi_{11}) \times \\ (\tilde{\alpha}^2\tilde{\chi}_1 + \tilde{\chi}_2) = H_{00}^i(-2\tilde{b}^2\tilde{\beta}) \times (\alpha^{10}\pi_1 - \alpha^9\pi_2 + \alpha^8\pi_3 \\ - \alpha^7\pi_4 + \alpha^6\pi_5 - \alpha^5\pi_6 + \alpha^4\pi_7 - \alpha^3\pi_8 + \alpha^2\pi_9 - \alpha\pi_{10} + \pi_{11}). \end{array} \right. \quad (2.2.5)$$

Upon adding (2.2.4) and (2.2.5), the result is expressed as follows:

$$\left\{ \begin{array}{l} (\alpha^{11}\chi_1 + \alpha^9\chi_3 + \alpha^7\chi_5 + \alpha^5\chi_7 + \alpha^3\chi_9 + \alpha\chi_{11}) \times (2\tilde{b}^2\tilde{\beta}) \\ + (\alpha^9\pi_2 + \alpha^7\pi_4 + \alpha^5\pi_6 + \alpha^3\pi_8 + \alpha\pi_{10}) \times (\tilde{\alpha}^2\tilde{\chi}_1 + \tilde{\chi}_2) \\ = H_{00}^i(2\tilde{b}^2\tilde{\beta}) \times (\alpha^9\pi_2 + \alpha^7\pi_4 + \alpha^5\pi_6 + \alpha^3\pi_8 + \alpha\pi_{10}). \end{array} \right. \quad (2.2.6)$$

Subtracting (2.2.5) from (2.2.4), we get

$$\left\{ \begin{array}{l} (\alpha^{10}\chi_2 + \alpha^8\chi_4 + \alpha^6\chi_6 + \alpha^4\chi_8 + \alpha^2\chi_{10} + \chi_{12}) \times (2\tilde{b}^2\tilde{\beta}) + \\ (\alpha^{10}\pi_1 + \alpha^8\pi_3 + \alpha^6\pi_5 + \alpha^4\pi_7 + \alpha^2\pi_9 + \pi_{11}) \times (\tilde{\alpha}^2\tilde{\chi}_1 + \tilde{\chi}_2) \\ = H_{00}^i(2\tilde{b}^2\tilde{\beta}) \times (\alpha^{10}\pi_1 + \alpha^8\pi_3 + \alpha^6\pi_5 + \alpha^4\pi_7 + \alpha^2\pi_9 + \pi_{11}). \end{array} \right. \quad (2.2.7)$$

According to equation (2.2.6), we have the following calculation:

$$\left\{ \begin{array}{l} (\alpha^{10}\chi_1 + \alpha^8\chi_3 + \alpha^6\chi_5 + \alpha^4\chi_7 + \alpha^2\chi_9 + \chi_{11}) \times (2\tilde{b}^2\tilde{\beta}) \\ + (\alpha^8\pi_2 + \alpha^6\pi_4 + \alpha^4\pi_6 + \alpha^2\pi_8 + \pi_{10}) \times (\tilde{\alpha}^2\tilde{\chi}_1 + \tilde{\chi}_2) \\ = H_{00}^i(2\tilde{b}^2\tilde{\beta}) \times (\alpha^8\pi_2 + \alpha^6\pi_4 + \alpha^4\pi_6 + \alpha^2\pi_8 + \pi_{10}). \end{array} \right. \quad (2.2.8)$$

$(\alpha^8\pi_2 + \alpha^6\pi_4 + \alpha^4\pi_6 + \alpha^2\pi_8 + \pi_{10}) \times (\tilde{\alpha}^2\tilde{\chi}_1 + \tilde{\chi}_2)$ divided by $\tilde{\beta}$, as per equation (2.2.8).

It is also possible to divide the formula $\tilde{\alpha}^2\tilde{\chi}_1\alpha^8\pi_2$ by $\tilde{\beta}$. With regard to α and $\tilde{\alpha}$, $\beta = \mu\tilde{\beta}$ and $\tilde{\beta}$ are prime. $\tilde{\chi}_1 := \tilde{b}^2\tilde{s}_0^i - \tilde{b}^i\tilde{s}_0$, therefore can be divided by $\tilde{\beta}$. Hence, there exists a scalar function $\psi^i(x)$ such that

$$\tilde{b}^2\tilde{s}_0^i - \tilde{b}^i\tilde{s}_0 = \tilde{\beta}\psi^i. \quad (2.2.9)$$

Contracting (2.2.9) by $\tilde{y}_i := \tilde{a}_{ij}y^j$, we obtained $\psi^i(\tilde{y}_i) = -\tilde{s}_0$. As $\tilde{y}_i$ is an arbitray vector, we get $\psi^i(x) = -\tilde{s}^i$. Then we have,

$$\tilde{s}_{ij} = \frac{1}{\tilde{b}^2} [\tilde{b}_i\tilde{s}_j - \tilde{b}_j\tilde{s}_i], \quad (2.2.10)$$

given that $\tilde{b}^2 \neq 0$. Lemma (2.1.1) states that the Douglas metric is $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$. F and $\tilde{F}$ are both Douglas metrics because they have the same Douglas tensor.

Now we are able to claim the following theorem

Theorem 2.2.2. *Let $F = \sqrt{\alpha(\alpha + \beta)}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ denote a square root (α, β) -metric and a Kropina metric, respectively, defined on an n -dimensional manifold M , where $n \geq 3$, α and $\tilde{\alpha}$ are Riemannian metrics, and $\beta, \tilde{\beta}$ are non-zero 1-forms. Then F is projectively related to $\tilde{F}$ if and only if the following conditions are satisfied:*

$$(i) \quad G_{\alpha}^i = \tilde{G}_{\alpha}^i + \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2 \tilde{s}_i + r_{00} \tilde{b}^i] + \theta y^i - \frac{2\tau\alpha^2((3\beta^2 - (b^2 - 4)\alpha^2)b^i}{6\beta^2 + 12\alpha\beta + 8\alpha^2 - 2b^2\alpha^2},$$

$$(ii) \quad b_{i|j} = 2\tau \{(b^2 - 4)a_{ij} - 3b_i b_j\},$$

$$(iii) \quad \tilde{s}_{ij} = \frac{1}{\tilde{b}^2} [\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i].$$

The square root (α, β) -metric $F = \sqrt{\alpha(\alpha + \beta)}$ is a Douglas metric if and only if $b_{i|j} = 2\tau \{(b^2 - 4)a_{ij} - 3b_i b_j\}$ holds for a scalar $\tau = \tau(x)$. It is known that the Kropina metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ is a Douglas metric if and only if $\tilde{s}_{ij} = \frac{1}{\tilde{b}^2} [\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i]$.

Proof. We establish by demonstrating the necessity. Invariance under projective change between two Finsler metrics is the definition of the Douglas tensor. Proposition (2.2.1) leads us to the conclusion that both $\tilde{F}$ and F are Douglas metrics. Kropina's metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ is a Douglas metric if and only if

$$\tilde{s}_{ij} = \frac{1}{\tilde{b}^2} [\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i]. \quad (2.2.11)$$

In the present investigation, we demonstrated that square root (α, β) -metric is Douglas metric if and only if

$$b_{i|j} = 2\tau \{(b^2 - 4)a_{ij} - 3b_i b_j\}. \quad (2.2.12)$$

For a scalar function $\tau = \tau(x)$, where $b_{i|j}$ represents the coefficients of the covariant derivatives of β with respect to α . In this case, β is closed. After entering (2.2.12) and (2.2.1) into (2.1.3), we obtain

$$\left\{ \begin{aligned} G^i &= G_{\alpha}^i + \frac{2\tau\alpha^2((3\beta^2 - (b^2 - 4)\alpha^2)b^i}{6\beta^2 + 12\alpha\beta + 8\alpha^2 - 2b^2\alpha^2} + \frac{4\tau(\alpha + \beta)((b^2 - 4)\alpha^2 - 3\beta^2)y^i}{6\beta^2 + 12\alpha\beta + 8\alpha^2 - 2b^2\alpha^2}. \end{aligned} \right. \quad (2.2.13)$$

However, substituting (2.2.11) and (2.2.2) into (2.1.3), we obtain

$$\tilde{G}^i = \tilde{G}_{\tilde{\alpha}}^i - \frac{1}{2\tilde{b}^2} \left[-\tilde{\alpha}^2 \tilde{s}^i + (2\tilde{s}_0 y^i - \tilde{r}_{00} \tilde{b}^i) + \frac{2\tilde{r}_{00} \tilde{\beta} y^i}{\tilde{\alpha}^2} \right]. \quad (2.2.14)$$

As F is projectively connected to $\tilde{F}$ again, there is a scalar function $P = P(x, y)$ on $TM \setminus \{0\}$ such that [118]

$$G^i = \tilde{G}^i + P y^i \quad (2.2.15)$$

. We obtain the following from (2.2.13), (2.2.14) and (2.2.15);

$$\left\{ \begin{aligned} & \left[P - \frac{4\tau(\alpha + \beta)((b^2 - 4)\alpha^2 - 3\beta^2)}{6\beta^2 + 12\alpha\beta + 8\alpha^2 - 2b^2\alpha^2} - \frac{1}{\tilde{b}^2} \left(\tilde{s}_0 + \frac{\tilde{r}_{00}\tilde{\beta}}{\tilde{\alpha}^2} \right) \right] y^i \\ & = G_{\alpha}^i + \frac{2\tau\alpha^2((3\beta^2 - (b^2 - 4)\alpha^2)b^i)}{6\beta^2 + 12\alpha\beta + 8\alpha^2 - 2b^2\alpha^2} - \tilde{G}_{\tilde{\alpha}}^i - \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2 \tilde{s}^i + \tilde{r}_{00} \tilde{b}^i]. \end{aligned} \right. \quad (2.2.16)$$

The right-hand side of the above equation is quadratic in y . Thus, $\theta = \theta_i(x)y^i$ on M must be of the form

$$P - \frac{4\tau(\alpha + \beta)((b^2 - 4)\alpha^2 - 3\beta^2)}{6\beta^2 + 12\alpha\beta + 8\alpha^2 - 2b^2\alpha^2} - \frac{1}{\tilde{b}^2} \left(\tilde{s}_0 + \frac{\tilde{r}_{00}\tilde{\beta}}{\tilde{\alpha}^2} \right) = \theta. \quad (2.2.17)$$

Then, we have

$$\left\{ \begin{aligned} G_{\alpha}^i &= \tilde{G}_{\tilde{\alpha}}^i - \frac{2\tau\alpha^2((3\beta^2 - (b^2 - 4)\alpha^2)b^i)}{6\beta^2 + 12\alpha\beta + 8\alpha^2 - 2b^2\alpha^2} + \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2 \tilde{s}^i + \tilde{r}_{00} \tilde{b}^i] + \theta y^i. \end{aligned} \right. \quad (2.2.18)$$

We conclude the evidence of necessity with (2.2.11), (2.2.12), and (2.2.18).

On the other hand, using (2.2.1) to put (2.2.12) into (2.1.3) results in (2.1.30). (2.2.11) yileds (2.2.2) are plugged into (2.1.3) and (2.2.14). These come from (2.2.13) and (2.2.18):

$$\left\{ \begin{aligned} G^i &= \tilde{G}_{\tilde{\alpha}}^i + \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2 \tilde{s}^i + \tilde{r}_{00} \tilde{b}^i] + y^i \left[\theta + \frac{4\tau(\alpha + \beta)((b^2 - 4)\alpha^2 - 3\beta^2)}{6\beta^2 + 12\alpha\beta + 8\alpha^2 - 2b^2\alpha^2} \right]. \end{aligned} \right. \quad (2.2.19)$$

Using equation (2.2.14) to obtain the value of $\tilde{G}_{\tilde{\alpha}}^i$, we obtain

$$\left\{ \begin{aligned} G^i &= \tilde{G}^i + \left\{ \left(\frac{1}{2\tilde{b}^2} \left[2\tilde{s}_0 + \frac{2\tilde{r}_{00}\tilde{\beta}}{\tilde{\alpha}^2} \right] \right) \right. \\ &\quad \left. + \left(\theta + \frac{4\tau(\alpha + \beta)((b^2 - 4)\alpha^2 - 3\beta^2)}{6\beta^2 + 12\alpha\beta + 8\alpha^2 - 2b^2\alpha^2} \right) \right\} y^i. \end{aligned} \right. \quad (2.2.20)$$

From equation (2.2.15), we get

$$P = \theta + \frac{4\tau(\alpha + \beta)((b^2 - 4)\alpha^2 - 3\beta^2)}{6\beta^2 + 12\alpha\beta + 8\alpha^2 - 2b^2\alpha^2} + \frac{1}{2\tilde{b}^2} \left[2\tilde{s}_0 + \frac{2\tilde{r}_{00}\tilde{\beta}}{\tilde{\alpha}^2} \right]. \quad (2.2.21)$$

i.e. F is projectively related to $\tilde{F}$. Hence complete the proof of Theorem (2.2.2).

□

□

Corollary 2.2.3. *Let $F = \sqrt{\alpha(\alpha + \beta)}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be two (α, β) -metrics on a manifold M of dimension $n > 2$, where α and $\tilde{\alpha}$ are Riemannian metrics, and $\beta = b_i y^i$ and $\tilde{\beta}$ are non-zero 1-forms. Then F is projectively related to $\tilde{F}$ if and only if both are Douglas metrics and the spray coefficients of α and $\tilde{\alpha}$ satisfy the following relation:*

$$G^i_\alpha = \tilde{G}^i_{\tilde{\alpha}} + \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2 \tilde{s}_i + r_{00} \tilde{b}^i] + \theta y^i - \frac{2\tau\alpha^2((3\beta^2 - (b^2 - 4)\alpha^2)b^i)}{6\beta^2 + 12\alpha\beta + 8\alpha^2 - 2b^2\alpha^2}, \quad (2.2.22)$$

where $\tilde{b}^i := \tilde{a}^{ij}\tilde{b}_j$, $\tau = \tau(x)$ is a scalar function and $\theta = \theta_i y^i$ is a 1-form on M .

2.2.1 Projective Change with Curvature Properties

Theorem 2.2.4. *Let $F = \sqrt{\alpha(\alpha + \beta)}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be two (α, β) -metrics on a manifold M with dimension $n > 2$, where α and $\tilde{\alpha}$ are two Riemannian metrics, $\beta = b_i y^i$ and $\tilde{\beta}$ are two non-zero 1-forms. Suppose that F has isotropic S -curvature. Then F is projectively related*

to $\tilde{F}$ if and only if the following conditions hold:

- (i) $G_\alpha^i = \tilde{G}_\alpha^i + \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2 \tilde{s}^i + \tilde{r}_{00} \tilde{b}^i] + \theta y^i$;
- (ii) β is parallel with respect to α , that is, $b_{i|j} = 0$;
- (iii) $\tilde{s}_{ij} = \frac{1}{\tilde{b}^2} [\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i]$.

Here, $b_{i|j}$ denotes the coefficients of the covariant derivatives of β with respect to α and $\tilde{b}^i := \tilde{\alpha}^{ij} \tilde{b}_j$, $\tilde{b} := ||\tilde{\beta}||_{\tilde{\alpha}}$ and $\theta = \theta_i y^i$ is a 1-form on M .

Proof. It is evident from Theorem (2.2.2) that sufficiency exists. If $\tilde{F}$ is projectively related to F , then according to Theorem (2.2.2),

$$b_{i|j} = 2\tau \{ (b^2 - 4)a_{ij} - 3b_i b_j \}.$$

$\tau = \tau(x)$ is scalar function. By contracting above equation with y^i and y^j yields

$$r_{00} = 2\tau \{ (b^2 - 4)\alpha^2 - 3\beta^2 \}. \quad (2.2.23)$$

In accordance with Lemma (2.1.5), $r_{00} = s_0 = 0$. if F has isotropic S-curvature.

We obtain through putting $r_{00} = s_0 = 0$ in equation (2.2.23).

$$(b^2 - 4)\alpha^2 - 3\beta^2 = 0,$$

provided $\tau \neq 0$. That is

$$(b^2 - 4)a_{ij} - 3b_i b_j = 0.$$

Using a^{ij} to contract the equation above results in $-4n + (n - 3)b^2 = 0$, which is not feasible. Consequently, since $n \geq 2$ makes $\tau = 0$ impossible, we end the proof by adding Theorem (2.2.2).

We are aware that the isotropic mean Berwald curvature and the Finsler metric for isotropic Berwald curvature must be identical. Lemma (2.1.5) leads to the theorem, which is derived assuming that F possesses isotropic Berwald curvature. $\square$

Theorem 2.2.5. *Let $F = \sqrt{\alpha(\alpha + \beta)}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be two (α, β) -metrics on a manifold M with dimension $n > 2$, where α and $\tilde{\alpha}$ are two Riemannian metrics, $\beta = b_i y^i$ and $\tilde{\beta}$ are two non-zero 1-forms. Suppose that F has isotropic Berwald curvature. Then F is projectively related to $\tilde{F}$ if and only if the following conditions hold:*

1. $G_\alpha^i = \tilde{G}_\alpha^i + \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2 \tilde{s}^i + \tilde{r}_{00} \tilde{b}^i] + \theta y^i$;
2. β is parallel with respect to α , that is, $b_{i|j} = 0$;
3. $\tilde{s}_{ij} = \frac{1}{\tilde{b}^2} [\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i]$, where $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α and $\tilde{b}^i := \tilde{a}^{ij} \tilde{b}_j$, $\tilde{b} := \|\tilde{\beta}\|_\alpha$ and $\theta = \theta_i y^i$ is a 1-form on M .

Theorem 2.2.6. *Let $F = \sqrt{\alpha(\alpha + \beta)}$ be projectively equivalent to $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ and $\tilde{F}$ has isotropic Berwald curvature. Then F has isotropic Berwald curvature if and only if F has isotropic S -curvature.*

Proof. If F has an isotropic Berwald curvature, then its mean Berwald curvature is also isotropic. Therefore, F has an isotropic S -curvature according to Lemma (2.1.5). This establishes a prerequisite for demonstrating sufficient. (2.1.1) is true since F and $\tilde{F}$ are projectively equal. Assume that $S = (n+1)c(x)F$ is the isotropic S -curvature of F . We know that F is of isotropic mean Berwald curvature by Lemma (2.1.5), which means that $E_{ij} = \frac{n+1}{2} c F_{y^i y^j}$.

Given that $\tilde{F}$ has isotropic Berwald curvature, then [118]

$$\tilde{B}_{jkl}^i = \tilde{c} \left(\tilde{F}_{y^j y^k} \delta_l^i + \tilde{F}_{y^j y^l} \delta_k^i + \tilde{F}_{y^k y^l} \delta_j^i + \tilde{F}_{y^j y^k y^l} y^i \right).$$

Hence, by the definition of the mean Berwald tensor, it follows from (2.1.1) that $c F_{y^i y^j} =$

$\tilde{c}\tilde{F}_{y^i y^j} + P_{y^i y^j}$, which gives [118]

$$cF_{y^i y^j y^k} = \tilde{c}\tilde{F}_{y^i y^j y^k} + P_{y^i y^j y^k}.$$

Now we have

$$\left\{ \begin{aligned} B_{jkl}^i &= \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} \\ &= \tilde{B}_{jkl}^i + (P_{y^j y^k} \delta_l^i + P_{y^j y^l} \delta_k^i + P_{y^k y^l} \delta_j^i + P_{y^j y^k y^l} y^i) \\ &= \tilde{c}(\tilde{F}_{y^j y^k} \delta_l^i + \tilde{F}_{y^j y^l} \delta_k^i + \tilde{F}_{y^k y^l} \delta_j^i + \tilde{F}_{y^j y^k y^l} y^i) \\ &\quad + (P_{y^j y^k} \delta_l^i + P_{y^j y^l} \delta_k^i + P_{y^k y^l} \delta_j^i + P_{y^j y^k y^l} y^i) \\ &= c(F_{y^j y^k} \delta_l^i + F_{y^j y^l} \delta_k^i + F_{y^k y^l} \delta_j^i + F_{y^j y^k y^l} y^i). \end{aligned} \right.$$

This suggests that the Berwald curvature of F is isotropic. We finish the proof. The theorem might be obtained using the methods mentioned above.

□

Theorem 2.2.7. *Let $F = \sqrt{\alpha(\alpha + \beta)}$ be projectively equivalent to $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$, and suppose F has isotropic Berwald curvature. Then $\tilde{F}$ has isotropic Berwald curvature if and only if $\tilde{F}$ has isotropic S -curvature.*

CHAPTER 3

Finsler Space with (α, β) -Metric

In this chapter, we analyze Berwald spaces, Douglas spaces, and weakly-Berwald spaces within the context of a Finsler space characterized by an exponential (α, β) -metric expressed as $F(\alpha, \beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$, and investigate the various conditions under which this metric qualifies as a Berwald space, Douglas space, and weakly-Berwald space.

3.1 Introduction

In 1928, L. Berwald [19] introduced the notion of an affinely connected Finsler space, in which the connection coefficients G_{jk}^i of the Berwald connection $B\Gamma$ are functions of the position coordinates x^i alone. He also defined the equation of geodesics as

$$\frac{d^2 x^i}{ds^2} + G_{jk}^i(x) \frac{dx^j}{ds} \frac{dx^k}{ds} = 0.$$

Definition 3.1.1. *A Finsler space is called a Berwald space if the connection coefficients G_{jk}^i of $B\Gamma$ are functions of position x^i alone in any coordinate system.*

The notion of a Douglas space was introduced by S. Bacsó and M. Matsumoto [17] as a generalization of a Berwald space from the point of view of geodesic equations. It is remarkable that a Finsler space is a Douglas space if and only if the Douglas tensor vanishes identically. Afterwards, the conditions for Finsler spaces to be Berwald spaces and Douglas spaces with different special kinds of (α, β) -metrics have been studied by many authors [70, 80, 149].

The concept of a weakly-Berwald space was introduced by Matsumoto [79] as a generalization of a Berwald space, i.e., a Finsler space with an (α, β) -metric is a weakly-Berwald space if $B_m^m = \frac{\partial B^m}{\partial y^m}$ is a one-form [75], i.e., B_m^m is a homogeneous polynomial in (y^i) of

degree one. He also investigated that a Finsler space with an (α, β) -metric is a weakly-Berwald space if B_m^m is a homogeneous polynomial in (y^i) of degree two. Further, several authors [17, 75, 151] have studied various important and interesting properties of weakly-Berwald spaces.

3.1.1 Preliminaries

In this chapter, we consider an n -dimensional Finsler space $F^n = \{M^n, F(\alpha, \beta)\}$, that is, a pair consisting of an n -dimensional differentiable manifold M^n equipped with a fundamental function F as a special Finsler space with the metric

$$F(\alpha, \beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}, \quad (3.1.1)$$

Differentiating equation (3.1.1) partially with respect to α and β are given by

$$\left\{ \begin{array}{l} F_\alpha = \frac{\alpha^2 e^{\frac{\beta}{\alpha}} - \alpha \beta e^{\frac{\beta}{\alpha}} + \beta^2 e^{-\frac{\beta}{\alpha}}}{\alpha^2}, F_\beta = \frac{\alpha e^{\frac{\beta}{\alpha}} + \alpha e^{-\frac{\beta}{\alpha}} - \beta e^{-\frac{\beta}{\alpha}}}{\alpha^2} \\ F_{\alpha\alpha} = \frac{\alpha \beta^2 e^{\frac{\beta}{\alpha}} - 2\alpha \beta^2 e^{-\frac{\beta}{\alpha}} + \beta^3 e^{-\frac{\beta}{\alpha}}}{\alpha^4}, F_{\beta\beta} = \frac{\alpha e^{\frac{\beta}{\alpha}} - 2\alpha e^{-\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}}{\alpha^2} \\ F_{\alpha\beta} = \frac{-\alpha \beta e^{\frac{\beta}{\alpha}} + 2\alpha \beta e^{-\frac{\beta}{\alpha}} - \beta^2 e^{-\frac{\beta}{\alpha}}}{\alpha^3}, F_{\beta\alpha} = \frac{-\alpha \beta e^{\frac{\beta}{\alpha}} + 2\alpha \beta e^{-\frac{\beta}{\alpha}} - \beta^2 e^{-\frac{\beta}{\alpha}}}{\alpha^3}, \\ F_{\alpha\alpha\alpha} = \frac{-3\alpha^2 \beta^2 e^{\frac{\beta}{\alpha}} - \alpha \beta^3 e^{\frac{\beta}{\alpha}} + 6\alpha^2 \beta^2 e^{-\frac{\beta}{\alpha}} - 6\alpha \beta^3 e^{-\frac{\beta}{\alpha}} + \beta^4 e^{-\frac{\beta}{\alpha}}}{\alpha^6}, \\ F_{\alpha\alpha\beta} = \frac{2\alpha^2 \beta e^{\frac{\beta}{\alpha}} + \alpha \beta^2 e^{\frac{\beta}{\alpha}} - 4\alpha^2 \beta e^{-\frac{\beta}{\alpha}} + 5\alpha \beta^2 e^{-\frac{\beta}{\alpha}} - \beta^3 e^{-\frac{\beta}{\alpha}}}{\alpha^5}. \end{array} \right. \quad (3.1.2)$$

According to [86], if $\beta^2 F_\alpha + \alpha \gamma^2 F_{\alpha\alpha} \neq 0$, where $\gamma^2 = b^2 \alpha^2 - \beta^2$, then the function $G^i(x, y)$ can be expressed as follows:

$$2G^i = \gamma_{00}^i + 2B^i,$$

$$B^i = \frac{\alpha F_\beta}{F_\alpha} s_0^i + C^* \left\{ \frac{\beta F_\beta}{\alpha F} y^i - \frac{\alpha F_{\alpha\alpha}}{F_\alpha} \left(\frac{1}{\alpha} y^i - \frac{\alpha}{\beta} b^i \right) \right\}, \quad (3.1.3)$$

where $F_\alpha = \frac{\partial F}{\partial \alpha}$, $F_\beta = \frac{\partial F}{\partial \beta}$, $F_{\alpha\alpha} = \frac{\partial^2 F}{\partial \alpha \partial \alpha}$, the subscript 0 denotes the contraction with y^i , and we put

$$C^* = \frac{\alpha\beta(r_{00}F_\alpha - 2s_0\alpha F_\beta)}{2(\beta^2 F_\alpha + \alpha\gamma^2 F_{\alpha\alpha})}. \quad (3.1.4)$$

We shall denote the homogeneous polynomials in (y^i) of degree r by $hp(r)$ for brevity. For example, γ'_{00} is $hp(2)$.

From equation (3.1.4), the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ of F^n with an (α, β) -metric is given by

$$\begin{aligned} G^i_j &= \dot{\partial}_j G^i = \gamma^i_{0j} + B^i_j, \\ G^i_{jk} &= \dot{\partial}_k G^i_j = \gamma^i_{jk} + B^i_{jk}, \end{aligned}$$

where we put $B^i_j = \dot{\partial}_j B^i$ and $B^i_{jk} = \dot{\partial}_k B^i_j$. Here, $B^i(x, y)$ is called the difference vector [86]. In view of [86], B^i_{jk} is determined by

$$F_\alpha B^t_{ji} y^j y_t + \alpha F_\beta (B^t_{ji} b_t - b_{j;i}) y^j = 0, \quad (3.1.5)$$

where y_k is equal to $a_{ik} y^i$.

A Finsler space F^n with an (α, β) -metric is a Douglas space if and only if $B^{ij} = B^i y^j - B^j y^i$ is $hp(3)$ [17]. From the latter of equation (3.1.4), B^{ij} is written as follows:

$$B^{ij} = \frac{\alpha F_\beta}{F_\alpha} (s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 F_{\alpha\alpha}}{\beta F_\alpha} C^* (b^i y^j - b^j y^i). \quad (3.1.6)$$

Lemma 3.1.2. [122] Suppose that $\alpha^2 \equiv 0 \pmod{\beta}$, meaning that $a_{ij}(x)y^i y^j$ admits $b_i(x)y^i$ as a factor. Under this assumption, the dimension reduces to two and b^2 becomes zero. Consequently, there exists a quantity $\delta = d_i(x)y^i$ satisfying the conditions $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.

In this chapter, the symbol $(;)$ stands for h -covariant derivation with respect to the Riemannian connection in the associated Riemannian space (M^n, α) and $\gamma^i_{jk}(x)$ stands for the

Christoffel symbols in the space (M^n, α) . We use the following notations [75].

$$r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i}), \quad s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}), \quad r_j^i = a^{ir} r_{rj}, \quad s_j^i = a^{ir} s_{rj}, \quad r_j = b_r r_j^r,$$

$$s_j = b_r s_j^r, \quad b^i = a^{ir} b_r, \quad b^2 = a^{rs} b_r b_s.$$

In what follows, we analyze the functions $G^i(x, y)$ of F^n with an (α, β) -metric. Based on [75], these are presented in the following form:

$$2G^m = \gamma_{00}^m + 2B^m,$$

$$B^m = \left(\frac{E^*}{\alpha} \right) y^m + \left(\frac{\alpha F_\beta}{F \alpha} \right) s_0^m - \left(\frac{\alpha F_{\alpha\alpha}}{F \alpha} \right) C^* \left\{ \left(\frac{y^m}{\alpha} \right) - \left(\frac{\alpha}{\beta} \right) b^m \right\}, \quad (3.1.7)$$

where,

$$E^* = \left(\frac{\beta F_\beta}{F} \right) C^*, \quad C^* = \frac{\alpha \beta (r_{00} F_\alpha - 2s_0 \alpha F_\beta)}{2(\beta^2 F_\alpha + \alpha \gamma^2 F_{\alpha\alpha})}, \quad \gamma^2 = b^2 \alpha^2 - \beta^2, \quad (3.1.8)$$

$$\left\{ \begin{array}{l} B_m^m = \left\{ \dot{\partial}_m \left(\frac{\beta F_\beta}{\alpha F} \right) y^m + \frac{n \beta F_\beta}{\alpha F} - \dot{\partial}_m \left(\frac{\alpha F_{\alpha\alpha}}{F \alpha} \right) \left(\frac{\beta y^m - \alpha^2 b^m}{\alpha \beta} \right) \right\} C^* \\ - \frac{\alpha F_{\alpha\alpha}}{F \alpha} \left\{ \dot{\partial}_m \left(\frac{1}{\alpha} \right) y^m + \left(\frac{1}{\alpha} \right) \delta_m^m - \dot{\partial}_m \left(\frac{\alpha}{\beta} \right) b^m \right\} C^* + \left(\frac{\beta F_\alpha F_\beta - \alpha F F_\alpha}{\alpha F F_\alpha} \right) \\ (\dot{\partial}_m C^*) y^m + \left(\frac{\alpha^2 F_{\alpha\alpha}}{\beta F_\alpha} \right) (\dot{\partial}_m C^*) b^m + \dot{\partial}_m \left(\frac{\alpha F_\beta}{F \alpha} \right) s_0^m. \end{array} \right.$$

Since $F = F(\alpha, \beta)$ is a positively homogeneous function of α and β of degree one, we have

$$F_\alpha \alpha + F_\beta \beta = F, \quad F_{\alpha\alpha} \alpha + F_{\alpha\beta} \beta = 0,$$

$$F_{\beta\alpha} \alpha + F_{\beta\beta} \beta = 0, \quad F_{\alpha\alpha\alpha} \alpha + F_{\alpha\alpha\beta} \beta = -F_{\alpha\alpha}.$$

Using the above and the homogeneity of (y^i) , we obtain

$$\dot{\partial}_m \left(\frac{\beta F_\beta}{\alpha F} \right) y^m = -\frac{\beta F_\beta}{\alpha F}, \quad (3.1.9)$$

$$\dot{\partial}_m \left(\frac{\alpha F_{\alpha\alpha}}{F\alpha} \right) \left(\frac{\beta y^m - \alpha^2 b^m}{\alpha\beta} \right) = \frac{\gamma^2}{(\beta F_\alpha)^2} \{F_\alpha F_{\alpha\alpha} + \alpha F_\alpha F_{\alpha\alpha\alpha} - \alpha (F_{\alpha\alpha})^2\}, \quad (3.1.10)$$

$$\dot{\partial}_m \left(\frac{1}{\alpha} \right) y^m + \left(\frac{1}{\alpha} \right) \delta_m^m - \dot{\partial}_m \left(\frac{\alpha}{\beta} \right) b^m = \frac{1}{\alpha\beta^2} \{\gamma^2 + (n-1)\beta^2\}, \quad (3.1.11)$$

$$(\dot{\partial}_m C^*) y^m = 2C^*, \quad (3.1.12)$$

$$\begin{cases} (\dot{\partial}_m C^*) b^m = \frac{1}{2\alpha\beta\Omega^2} [\Omega \{\beta(\gamma^2 + 2\beta^2)W + 2\alpha^2\beta^2 F_\alpha r_0 - \alpha\beta\gamma^2 F_{\alpha\alpha} r_{00} \\ - 2\alpha(\beta^3 F_\beta + \alpha^2\gamma^2 F_{\alpha\alpha}) s_0\} - \alpha^2\beta W \{2b^2\beta^2 F_\alpha - \gamma^4 F_{\alpha\alpha\alpha} - b^2\alpha\gamma^2 F_{\alpha\alpha}\}], \end{cases} \quad (3.1.13)$$

$$\dot{\partial}_m \left(\frac{\alpha F_\beta}{F\alpha} \right) s_0^m = \frac{\alpha^2 F F_{\alpha\alpha} s_0}{(\beta F_\alpha)^2}, \quad (3.1.14)$$

where,

$$W = (r_{00} F_\alpha - 2s_0 \alpha F_\beta), \Omega = (\beta^2 F_\alpha + \alpha\gamma^2 F_{\alpha\alpha}), \text{ given that } \Omega \neq 0. \quad (3.1.15)$$

$$Y_i = a_{ir} y^r, \quad s_{00} = 0, \quad b^r s_r = 0, \quad a^{ij} s_{ij} = 0.$$

Substituting (3.1.9), (3.1.10), (3.1.11), (3.1.12), (3.1.13) and (3.1.14) into (3.1.8), we have

$$\begin{cases} B_m^m = \frac{1}{2\alpha F(\beta F_\alpha)^2 \Omega^2} \left\{ 2\Omega^2 A C^* + 2\alpha F \Omega^2 B s_0 + \right. \\ \left. \alpha^2 F F_\alpha F_{\alpha\alpha} (C r_{00} + D s_0 + E r_0) \right\}, \end{cases} \quad (3.1.16)$$

$$\left\{ \begin{array}{l} A = (n+1)\beta^2 F_\alpha (\beta F_\alpha F_\beta - \alpha F F_{\alpha\alpha}) + \alpha \gamma^2 F \{ \alpha (F_{\alpha\alpha})^2 - 2F_\alpha F_{\alpha\alpha} - \alpha F_\alpha F_{\alpha\alpha\alpha} \}, \\ B = \alpha^2 F F_{\alpha\alpha}, \\ C = \beta \gamma^2 \{ -\beta^2 (F_\alpha)^2 + 2b^2 \alpha^3 F_\alpha F_{\alpha\alpha} - \alpha^2 \gamma^2 (F_{\alpha\alpha})^2 + \alpha^2 \gamma^2 F_\alpha F_{\alpha\alpha\alpha} \}, \\ D = 2\alpha \{ \beta^3 (\gamma^2 - \beta^2) F_\alpha F_\beta - \alpha^2 \beta^2 \gamma^2 F_\alpha F_{\alpha\alpha} \\ - 2\alpha \beta \gamma^2 (\gamma^2 + 2\beta^2) F_\beta F_{\alpha\alpha} - \alpha^3 \gamma^4 (F_{\alpha\alpha})^2 - \alpha^2 \beta \gamma^4 F_\beta F_{\alpha\alpha\alpha} \}, \\ E = 2\alpha^2 \beta^2 F_\alpha \Omega. \end{array} \right. \quad (3.1.17)$$

Thus, we have [75].

Theorem 3.1.3. *A Finsler space F^n equipped with an (α, β) -metric constitutes a weakly-Berwald space if and only if $G_m^m = \gamma_{0m}^m + B_m^m$, such that B_m^m remains a homogeneous polynomial of degree one in y^m . Here, B_m^m is expressed through (3.1.16) and (3.1.17), under the condition that $\Omega^2 \neq 0$.*

Lemma 3.1.4. [58] *If α^2 admits β as a factor, then the dimension reduces to two and $b^2 \neq 0$. Throughout this paper, we assume that the dimension exceeds two and $b^2 \neq 0$, that is, $\alpha^2 \not\equiv 0 \pmod{\beta}$.*

3.1.2 The Condition to be a Berwald space

In this section, we have obtained the condition for a Finsler space F^n with an exponential (α, β) -metric $F(\alpha, \beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$ to be a Berwald space.

From equations (3.1.2) and (3.1.5), we have

$$(\alpha^2 e^{\frac{\beta}{\alpha}} - \alpha \beta e^{\frac{\beta}{\alpha}} + \beta^2 e^{-\frac{\beta}{\alpha}}) B_{ji}^t y^j y_t + \alpha^2 (\alpha e^{\frac{\beta}{\alpha}} + \alpha e^{-\frac{\beta}{\alpha}} - \beta e^{-\frac{\beta}{\alpha}}) (B_{ji}^t b_t - b_{j;i}) y^j = 0. \quad (3.1.18)$$

Suppose that F^n is a Berwald space, i.e., $B_{jk}^i = B_{jk}^i(x)$. Separating equation (3.1.18) into rational and irrational parts in y^i , we get

$$(\alpha^2 e^{\frac{\beta}{\alpha}} + \beta^2 e^{-\frac{\beta}{\alpha}}) B_{ji}^t y^j y_t - (\alpha^2 \beta e^{-\frac{\beta}{\alpha}}) (B_{ji}^t b_t - b_{j;i}) y^j = 0, \quad (3.1.19)$$

and

$$(-\beta e^{\frac{\beta}{\alpha}})B_{ji}^t y^j y_t + (\alpha^2 e^{\frac{\beta}{\alpha}} + \alpha^2 e^{-\frac{\beta}{\alpha}})(B_{ji}^t b_t - b_{j;i})y^j = 0. \quad (3.1.20)$$

Equations (3.1.19) and (3.1.20) can be expressed as a system of homogeneous linear equations in matrix form: $AX = 0$,

$$\begin{bmatrix} \alpha^2 e^{\frac{\beta}{\alpha}} + \beta^2 e^{-\frac{\beta}{\alpha}} & -\alpha^2 \beta e^{-\frac{\beta}{\alpha}} \\ -\beta e^{\frac{\beta}{\alpha}} & \alpha^2 e^{-\frac{\beta}{\alpha}} + \alpha^2 e^{\frac{\beta}{\alpha}} \end{bmatrix} \begin{bmatrix} B_{ji}^t y^j y_t \\ (B_{ji}^t b_t - b_{j;i})y^j \end{bmatrix} = 0. \quad (3.1.21)$$

Where,

$$A = \begin{bmatrix} \alpha^2 e^{\frac{\beta}{\alpha}} + \beta^2 e^{-\frac{\beta}{\alpha}} & -\alpha^2 \beta e^{-\frac{\beta}{\alpha}} \\ -\beta e^{\frac{\beta}{\alpha}} & \alpha^2 e^{-\frac{\beta}{\alpha}} + \alpha^2 e^{\frac{\beta}{\alpha}} \end{bmatrix} \quad (3.1.22)$$

and

$$|A| = \alpha^4 e^{\frac{2\beta}{\alpha}} + \alpha^4 + \alpha^2 \beta^2 e^{-\frac{2\beta}{\alpha}} \neq 0. \quad (3.1.23)$$

Which implies

$$B_{ji}^t y^j y_t = 0, \quad (3.1.24)$$

and

$$(B_{ji}^t b_t - b_{j;i})y^j = 0. \quad (3.1.25)$$

Contracting equation (3.1.25) with y_t , we have

$$(B_{ji}^t b_t - b_{j;i})y^j y_t = 0, \quad (3.1.26)$$

$$\begin{cases} \Rightarrow B_{ji}^t b_t y^j y_t - b_{j;i} y^j y_t = 0 \\ \Rightarrow (B_{ji}^t y^j y_t) b_t - b_{j;i} y^j y_t = 0. \end{cases} \quad (3.1.27)$$

Using equation (3.1.24) in (3.1.27), we have

$$b_{j,i} = 0.$$

Conversely, if $b_{j,i} = 0$ in equation (3.1.25), then we have

$$B_{ji}^t = 0.$$

Thus, we have

Theorem 3.1.5. *A Finsler space with an exponential (α, β) -metric $F(\alpha, \beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$ is a Berwald space if and only if $b_{i,j} = 0$.*

3.1.3 The Condition to be a Douglas Space

The present section is devoted to finding the condition for a Finsler space F^n with an exponential (α, β) -metric (3.1.1) to be a Douglas space. In Finsler space F^n with metric (3.1.1), we have

$$C^* = \frac{\alpha^2 \beta}{2} \left\{ \frac{W}{\Omega} \right\},$$

where,

$$\begin{aligned} W &= r_{00}(\alpha^2 e^{\frac{\beta}{\alpha}} - \alpha \beta e^{\frac{\beta}{\alpha}} + \beta^2 e^{-\frac{\beta}{\alpha}}) - 2S_0 \alpha^2 (\alpha e^{\frac{\beta}{\alpha}} + \alpha e^{-\frac{\beta}{\alpha}} - \beta e^{-\frac{\beta}{\alpha}}), \\ \Omega &= \alpha \beta^2 (\alpha^2 e^{\frac{\beta}{\alpha}} - \alpha \beta e^{\frac{\beta}{\alpha}} + \beta^2 e^{-\frac{\beta}{\alpha}}) + \alpha^2 b^2 (\alpha \beta^2 e^{\frac{\beta}{\alpha}} - 2\alpha \beta^2 e^{-\frac{\beta}{\alpha}} + \beta^3 e^{-\frac{\beta}{\alpha}}) \\ &\quad - \beta^2 (\alpha \beta^2 e^{\frac{\beta}{\alpha}} - 2\alpha \beta^2 e^{-\frac{\beta}{\alpha}} + \beta^3 e^{-\frac{\beta}{\alpha}}) \neq 0. \end{aligned}$$

In view of [86], if $\Omega = \beta^2 F_\alpha + \alpha \gamma^2 F_{\alpha\alpha} \neq 0$, then the function $G^i(x, y)$ can be written in the following form:

$$2G^i = \gamma_{00}^i + 2B^i,$$

where,

$$B^i = \frac{\alpha F_\beta s_0^i}{F_\alpha} + C^* \left\{ \frac{\beta F_\beta y^i}{\alpha F} - \frac{\alpha F_{\alpha\alpha}}{F_\alpha} \left(\frac{y^i}{\alpha} - \frac{\alpha b^i}{\beta} \right) \right\}.$$

The vector $B^i(x, y)$ is referred as the difference vector. Therefore, B^{ij} is expressed as

$$B^{ij} = \frac{\alpha F_\beta}{F_\alpha} (s_0^i y^j - s_0^j y^i) + \frac{\alpha^2 F_{\alpha\alpha}}{\beta F_\alpha} C^* (b^i y^j - b^j y^i). \quad (3.1.28)$$

Substituting equation (3.1.2) into the above equation, we get

$$\left\{ \begin{aligned} & (2\alpha^5 \beta^3 e^{\frac{2\beta}{\alpha}} - 4\alpha^4 \beta^4 e^{\frac{2\beta}{\alpha}} + 8\alpha^3 \beta^5 + 2\alpha^5 \beta^3 b^2 e^{\frac{2\beta}{\alpha}} - 4\alpha^5 \beta^3 b^2 + 6\alpha^4 \beta^4 b^2 \\ & - 10\alpha^2 \beta^6 - 2\alpha^4 \beta^4 b^2 e^{\frac{2\beta}{\alpha}} + 2\alpha^2 \beta^6 e^{\frac{2\beta}{\alpha}} + 6\alpha \beta^7 e^{\frac{-2\beta}{\alpha}} - 4\alpha^3 \beta^5 b^2 e^{\frac{-2\beta}{\alpha}} \\ & + 2\alpha^2 \beta^6 b^2 e^{\frac{-2\beta}{\alpha}} - 2\beta^8 e^{\frac{-2\beta}{\alpha}}) B^{ij} + (-2\alpha^6 \beta^3 e^{\frac{2\beta}{\alpha}} + 2\alpha^5 \beta^4 e^{\frac{2\beta}{\alpha}} \\ & - 6\alpha^4 \beta^5 - 2\alpha^6 \beta^3 b^2 e^{\frac{2\beta}{\alpha}} + 2\alpha^6 \beta^3 b^2 + 2\alpha^4 \beta^5 e^{\frac{2\beta}{\alpha}} - 2\alpha^6 \beta^3 + 4\alpha^5 \beta^4 \\ & - 2\alpha^4 \beta^5 e^{\frac{-2\beta}{\alpha}} + 4\alpha^6 \beta^3 b^2 e^{\frac{-2\beta}{\alpha}} - 6\alpha^5 \beta^4 b^2 e^{\frac{-2\beta}{\alpha}} - 4\alpha^4 \beta^5 e^{\frac{-2\beta}{\alpha}} \\ & + 8\alpha^3 \beta^6 e^{\frac{-2\beta}{\alpha}} + 2\alpha^4 \beta^5 b^2 e^{\frac{-2\beta}{\alpha}} - 2\alpha^2 \beta^7 e^{\frac{-2\beta}{\alpha}}) (s_0^i y^j - s_0^j y^i) \\ & + r_{00} (-\alpha^5 \beta^3 e^{\frac{2\beta}{\alpha}} + \alpha^4 \beta^4 e^{\frac{2\beta}{\alpha}} + 2\alpha^5 \beta^3 - 3\alpha^4 \beta^4 \\ & + 2\alpha^3 \beta^5 e^{\frac{-2\beta}{\alpha}} - \alpha^2 \beta^6 e^{\frac{-2\beta}{\alpha}}) + s_0 (2\alpha^6 \beta^3 e^{\frac{2\beta}{\alpha}} \\ & - 2\alpha^6 \beta^3 - 4\alpha^6 \beta^3 e^{\frac{-2\beta}{\alpha}} + 6\alpha^5 \beta^4 e^{\frac{-2\beta}{\alpha}} \\ & - 2\alpha^4 \beta^5 e^{\frac{-2\beta}{\alpha}}) (b^i y^j - b^j y^i) = 0. \end{aligned} \right. \quad (3.1.29)$$

If F^n is a Douglas space, then B^{ij} are $hp(8)$. Separating the rational and irrational terms in equation (3.1.29), we obtain

$$\left\{ \begin{aligned} & (-4\alpha^4 \beta^4 e^{\frac{2\beta}{\alpha}} + 6\alpha^4 \beta^4 b^2 - 10\alpha^2 \beta^6 - 2\alpha^4 \beta^4 b^2 e^{\frac{2\beta}{\alpha}} + 2\alpha^2 \beta^6 e^{\frac{2\beta}{\alpha}} \\ & + 2\alpha^2 \beta^6 b^2 e^{\frac{-2\beta}{\alpha}} - 2\beta^8 e^{\frac{-2\beta}{\alpha}}) B^{ij} + (-2\alpha^6 \beta^3 e^{\frac{2\beta}{\alpha}} - 6\alpha^4 \beta^5 \\ & - 2\alpha^6 \beta^3 b^2 e^{\frac{2\beta}{\alpha}} + 2\alpha^6 \beta^3 b^2 + 2\alpha^4 \beta^5 e^{\frac{2\beta}{\alpha}} - 2\alpha^6 \beta^3 - 2\alpha^4 \beta^5 e^{\frac{-2\beta}{\alpha}} \\ & + 4\alpha^6 \beta^3 b^2 e^{\frac{-2\beta}{\alpha}} - 4\alpha^4 \beta^5 e^{\frac{-2\beta}{\alpha}} + 2\alpha^4 \beta^5 b^2 e^{\frac{-2\beta}{\alpha}} - 2\alpha^2 \beta^7 e^{\frac{-2\beta}{\alpha}}) \\ & \times (s_0^i y^j - s_0^j y^i) + r_{00} (\alpha^4 \beta^4 e^{\frac{2\beta}{\alpha}} - 3\alpha^4 \beta^4 - \alpha^2 \beta^6 e^{\frac{-2\beta}{\alpha}}) \\ & + s_0 (2\alpha^6 \beta^3 e^{\frac{2\beta}{\alpha}} - 2\alpha^6 \beta^3 - 4\alpha^6 \beta^3 e^{\frac{-2\beta}{\alpha}} - 2\alpha^4 \beta^5 e^{\frac{-2\beta}{\alpha}}) \\ & \times (b^i y^j - b^j y^i) = 0. \end{aligned} \right. \quad (3.1.30)$$

and

$$\left\{ \begin{aligned} & (2\alpha^4\beta^3e^{\frac{2\beta}{\alpha}} + 8\alpha^2\beta^5 + 2\alpha^4\beta^3b^2e^{\frac{2\beta}{\alpha}} - 4\alpha^4\beta^3b^2 + 6\beta^7e^{\frac{-2\beta}{\alpha}} \\ & - 4\alpha^2\beta^5b^2e^{\frac{-2\beta}{\alpha}})B^{ij} + (2\alpha^4\beta^4e^{\frac{2\beta}{\alpha}} + 4\alpha^4\beta^4 - 6\alpha^4\beta^4b^2e^{\frac{-2\beta}{\alpha}} \\ & + 8\alpha^2\beta^6e^{\frac{-2\beta}{\alpha}})(S_0^iy^j - S_0^jy^i) + r_{00}(-\alpha^4\beta^3e^{\frac{2\beta}{\alpha}} + 2\alpha^4\beta^3 \\ & + 2\alpha^2\beta^5e^{\frac{-2\beta}{\alpha}}) + 6s_0\alpha^4\beta^4e^{\frac{-2\beta}{\alpha}}(b^iy^j - b^jy^i) = 0. \end{aligned} \right. \quad (3.1.31)$$

Eliminating B^{ij} from equations (3.1.30) and (3.1.31), we have

$$A(s_0^iy^j - s_0^jy^i) + B(b^iy^j - b^jy^i) = 0, \quad (3.1.32)$$

where,

$$\left\{ \begin{aligned} A &= (2\alpha^4\beta^3e^{\frac{2\beta}{\alpha}} + 8\alpha^2\beta^5 + 2\alpha^4\beta^3b^2e^{\frac{2\beta}{\alpha}} - 4\alpha^4\beta^3b^2 + 6\beta^7e^{\frac{-2\beta}{\alpha}} \\ & - 4\alpha^2\beta^5b^2e^{\frac{-2\beta}{\alpha}}) \times (-2\alpha^4\beta^3e^{\frac{2\beta}{\alpha}} - 6\alpha^2\beta^5 - 2\alpha^4\beta^3b^2e^{\frac{2\beta}{\alpha}} \\ & + 2\alpha^4\beta^3b^2 + 2\alpha^2\beta^5e^{\frac{2\beta}{\alpha}} - 2\alpha^4\beta^3 - 2\alpha^2\beta^5e^{\frac{-2\beta}{\alpha}} + 4\alpha^4\beta^3b^2e^{\frac{-2\beta}{\alpha}} \\ & - 4\alpha^2\beta^5e^{\frac{-2\beta}{\alpha}} + 2\alpha^2\beta^5b^2e^{\frac{-2\beta}{\alpha}} - 2\beta^7e^{\frac{-2\beta}{\alpha}}) - (-4\alpha^4\beta^4e^{\frac{2\beta}{\alpha}} \\ & + 6\alpha^4\beta^4b^2 - 10\alpha^2\beta^6 - 2\alpha^4\beta^4b^2e^{\frac{2\beta}{\alpha}} + 2\alpha^2\beta^6e^{\frac{2\beta}{\alpha}} \\ & + 2\alpha^2\beta^6b^2e^{\frac{-2\beta}{\alpha}} - 2\beta^8e^{\frac{-2\beta}{\alpha}}) \times (2\alpha^2\beta^4e^{\frac{2\beta}{\alpha}} \\ & + 4\alpha^2\beta^4 - 6\alpha^2\beta^4b^2e^{\frac{-2\beta}{\alpha}} + 8\beta^6e^{\frac{-2\beta}{\alpha}}), \\ B &= (2\alpha^4\beta^3e^{\frac{2\beta}{\alpha}} + 8\alpha^2\beta^5 + 2\alpha^4\beta^3b^2e^{\frac{2\beta}{\alpha}} - 4\alpha^4\beta^3b^2 + 6\beta^7e^{\frac{-2\beta}{\alpha}} \\ & - 4\alpha^2\beta^5b^2e^{\frac{-2\beta}{\alpha}}) \times r_{00}(\alpha^2\beta^4e^{\frac{2\beta}{\alpha}} - 3\alpha^2\beta^4 - \beta^6e^{\frac{-2\beta}{\alpha}}) \\ & + s_0(2\alpha^4\beta^3e^{\frac{2\beta}{\alpha}} - 2\alpha^4\beta^3 - 4\alpha^4\beta^3e^{\frac{-2\beta}{\alpha}} - 2\alpha^2\beta^5e^{\frac{-2\beta}{\alpha}}) \\ & - (-4\alpha^4\beta^4e^{\frac{2\beta}{\alpha}} + 6\alpha^4\beta^4b^2 - 10\alpha^2\beta^6 - 2\alpha^4\beta^4b^2e^{\frac{2\beta}{\alpha}} + 2\alpha^2\beta^6e^{\frac{2\beta}{\alpha}} \\ & + 2\alpha^2\beta^6b^2e^{\frac{-2\beta}{\alpha}} - 2\beta^8e^{\frac{-2\beta}{\alpha}}) \times \{r_{00}(-\alpha^2\beta^3e^{\frac{2\beta}{\alpha}} + 2\alpha^2\beta^3 \\ & + 2\beta^5e^{\frac{-2\beta}{\alpha}}) + 6s_0\alpha^2\beta^4e^{\frac{-2\beta}{\alpha}}\}. \end{aligned} \right. \quad (3.1.33)$$

Contracting equation (3.1.32) with b_iy_j , we obtain

$$As_0\alpha^2 + B(b^2\alpha^2 - \beta^2) = 0. \quad (3.1.34)$$

The term in equation (3.1.34) which does not contain α^2 is $2\beta^{15}r_{00}$. Therefore, we have $hp(15)$ implies v_{15} such that

$$r_{00}\beta^{15} = \alpha^2 v_{15}. \quad (3.1.35)$$

Here, we divide our discussion into following the three cases:

- (1) $v_{15} = 0$,
- (2) $v_{15} \neq 0, \alpha^2 \not\equiv 0 \pmod{\beta}$,
- (3) $v_{15} \neq 0, \alpha^2 \equiv 0 \pmod{\beta}$.

Case 1: In case of $V_{15} = 0$; from (3.1.35), $r_{00} = 0$, and (3.1.34) is reduced to

$$s_0\{A + B_1(b^2\alpha^2 - \beta^2)\} = 0, \quad (3.1.36)$$

where,

$$\left\{ \begin{array}{l} B_1 = (2\alpha^4\beta^3e^{\frac{2\beta}{\alpha}} + 8\alpha^2\beta^5 + 2\alpha^4\beta^3b^2e^{\frac{2\beta}{\alpha}} - 4\alpha^4\beta^3b^2 + 6\beta^7e^{\frac{-2\beta}{\alpha}} \\ - 4\alpha^2\beta^5b^2e^{\frac{-2\beta}{\alpha}}) \times (2\alpha^2\beta^3e^{\frac{2\beta}{\alpha}} - 2\alpha^2\beta^3 - 4\alpha^2\beta^3e^{\frac{-2\beta}{\alpha}} - 2\beta^5e^{\frac{-2\beta}{\alpha}}) \\ - ((-4\alpha^4\beta^4e^{\frac{2\beta}{\alpha}} + 6\alpha^4\beta^4b^2 - 10\alpha^2\beta^6 - 2\alpha^4\beta^4b^2e^{\frac{2\beta}{\alpha}} \\ + 2\alpha^2\beta^6e^{\frac{2\beta}{\alpha}} + 2\alpha^2\beta^6b^2e^{\frac{-2\beta}{\alpha}} - 2\beta^8e^{\frac{-2\beta}{\alpha}}) \times (6\beta^4e^{\frac{-2\beta}{\alpha}}). \end{array} \right. \quad (3.1.37)$$

If $A + B_1(b^2\alpha^2 - \beta^2) = 0$ in equation (3.1.36), then the term of this equation which does not contain α^2 is $4\beta^{14}$. Therefore, there exists $hp(12)v_{12}$ such that $4\beta^{14} = \alpha^2 v_{12}$.

In this case, if $\alpha^2 \not\equiv 0 \pmod{\beta}$, then we have $v_{12} = 0$, which leads a contradiction. Therefore, $A + B_1(b^2\alpha^2 - \beta^2) \neq 0$. If $\alpha^2 \equiv 0 \pmod{\beta}$, then $A + B_1(b^2\alpha^2 - \beta^2) = 0$ is written as follows:

$$\left\{ \begin{array}{l} [(2\delta^2e^{\frac{2\beta}{\alpha}} + 8\beta\delta + 6\beta^2e^{\frac{-2\beta}{\alpha}}) \times (-2\delta^2e^{\frac{2\beta}{\alpha}} - 6\beta\delta + 2\beta\delta e^{\frac{2\beta}{\alpha}} \\ - 2\delta^2 - 2\delta\beta e^{\frac{-2\beta}{\alpha}} - 4\delta\beta e^{\frac{-2\beta}{\alpha}} - 2\beta^2e^{\frac{-2\beta}{\alpha}}) - (-4\beta\delta^2e^{\frac{2\beta}{\alpha}} \\ - 10\beta^2\delta + 2\beta^2\delta e^{\frac{2\beta}{\alpha}} - 2\beta^3e^{\frac{-2\beta}{\alpha}}) \times (2\delta e^{\frac{2\beta}{\alpha}} + 4\delta + 8\beta e^{\frac{-2\beta}{\alpha}})] \end{array} \right. \quad (3.1.38)$$

$$\left\{ \begin{array}{l} -[(2\delta^2 e^{\frac{2\beta}{\alpha}} + 8\beta\delta + 6\beta^2 e^{\frac{-2\beta}{\alpha}}) \times (2\delta\beta e^{\frac{2\beta}{\alpha}} - 2\beta\delta - 4\beta\delta e^{\frac{-2\beta}{\alpha}} \\ -2\beta^2 e^{\frac{-2\beta}{\alpha}}) - (-4\delta^2 e^{\frac{2\beta}{\alpha}} - 10\beta\delta + 2\beta\delta e^{\frac{2\beta}{\alpha}} - 2\beta^2 e^{\frac{-2\beta}{\alpha}}) \\ \times (6\beta^2 e^{\frac{-2\beta}{\alpha}})] = 0. \end{array} \right.$$

This is provided that $b^2 = 0$ and $\alpha^2 = \beta\delta$ by Lemma (3.1.2). The term of (3.1.38) which does not contain β is $-8\delta^4$. Therefore, there exists $hp(3)v_3$ such that $-8\delta^4 = \beta v_3$, which leads to a contradiction. Hence $A + B_1(b^2\alpha^2 - \beta^2) \neq 0$.

Therefore, in either case, whether $\alpha^2 \not\equiv 0 \pmod{\beta}$ or $\alpha^2 \equiv 0 \pmod{\beta}$, we see that $A + B_1(b^2\alpha^2 - \beta^2) \neq 0$. Thus $s_0 = 0$ from (3.1.36). Put $s_0 = 0$ and $r_{00} = 0$ in (3.1.32), we have

$$A(s_0^i y^j - s_0^j y^i) = 0. \quad (3.1.39)$$

If $A = 0$, then we have from (3.1.33)

$$\left\{ \begin{array}{l} A = (2\alpha^4\beta^3 e^{\frac{2\beta}{\alpha}} + 8\alpha^2\beta^5 + 2\alpha^4\beta^3 b^2 e^{\frac{2\beta}{\alpha}} - 4\alpha^4\beta^3 b^2 + 6\beta^7 e^{\frac{-2\beta}{\alpha}} \\ -4\alpha^2\beta^5 b^2 e^{\frac{-2\beta}{\alpha}}) \times (-2\alpha^4\beta^3 e^{\frac{2\beta}{\alpha}} - 6\alpha^2\beta^5 - 2\alpha^4\beta^3 b^2 e^{\frac{2\beta}{\alpha}} \\ + 2\alpha^4\beta^3 b^2 + 2\alpha^2\beta^5 e^{\frac{2\beta}{\alpha}} - 2\alpha^4\beta^3 - 2\alpha^2\beta^5 e^{\frac{-2\beta}{\alpha}} \\ + 4\alpha^4\beta^3 b^2 e^{\frac{-2\beta}{\alpha}} - 4\alpha^2\beta^5 e^{\frac{-2\beta}{\alpha}} + 2\alpha^2\beta^5 b^2 e^{\frac{-2\beta}{\alpha}} - 2\beta^7 e^{\frac{-2\beta}{\alpha}}) \\ -(-4\alpha^4\beta^4 e^{\frac{2\beta}{\alpha}} + 6\alpha^4\beta^4 b^2 - 10\alpha^2\beta^6 - 2\alpha^4\beta^4 b^2 e^{\frac{2\beta}{\alpha}} + 2\alpha^2\beta^6 e^{\frac{2\beta}{\alpha}} \\ + 2\alpha^2\beta^6 b^2 e^{\frac{-2\beta}{\alpha}} - 2\beta^8 e^{\frac{-2\beta}{\alpha}}) \times (2\alpha^2\beta^4 e^{\frac{2\beta}{\alpha}} + 4\alpha^2\beta^4 \\ - 6\alpha^2\beta^4 b^2 e^{\frac{-2\beta}{\alpha}} + 8\beta^6 e^{\frac{-2\beta}{\alpha}}) = 0. \end{array} \right. \quad (3.1.40)$$

The term in (3.1.39) that does not contain α^2 is $4\beta^{14}$. Thus, there exists $hp(12)v_{12}$ such that $4\beta^{14} = \alpha^2 v_{12}$. From this equation, we have $v_{12} = 0$, which leads to a contradiction. Hence $A \neq 0$. Thus, we have from (3.1.39)

$$s_0^i y^j - s_0^j y^i = 0. \quad (3.1.41)$$

Transvecting (3.1.41) by y_j gives $s_0^i = 0$. Finally, $r_{ij} = s_{ij} = 0$ is concluded, that is, $b_{i,j} = 0$.

Case 2 : If $v_{15} \neq 0$ and $\alpha^2 \not\equiv 0 \pmod{\beta}$, then (3.1.36) shows that there exists a function $h = h(x)$ satisfying

$$r_{00} = h\alpha^2. \quad (3.1.42)$$

Substituting (3.1.42) into (3.1.34), we have

$$\left\{ \begin{array}{l} As_0 + [(2\alpha^4\beta^3e^{\frac{2\beta}{\alpha}} + 8\alpha^2\beta^5 + 2\alpha^4\beta^3b^2e^{\frac{2\beta}{\alpha}} - 4\alpha^4\beta^3b^2 \\ + 6\beta^7e^{\frac{-2\beta}{\alpha}} - 4\alpha^2\beta^5b^2e^{\frac{-2\beta}{\alpha}}) \times (h\alpha^2\beta^4e^{\frac{2\beta}{\alpha}} - 3h\alpha^2\beta^4 - h\beta^6e^{\frac{-2\beta}{\alpha}} \\ + 2s_0\alpha^2\beta^3e^{\frac{2\beta}{\alpha}} - 2s_0\alpha^2\beta^3 - 4s_0\alpha^2\beta^3e^{\frac{-2\beta}{\alpha}} - 2s_0\beta^5e^{\frac{-2\beta}{\alpha}}) \\ - ((-4\alpha^4\beta^4e^{\frac{2\beta}{\alpha}} + 6\alpha^4\beta^4b^2 - 10\alpha^2\beta^6 - 2\alpha^4\beta^4b^2e^{\frac{2\beta}{\alpha}} + 2\alpha^2\beta^6e^{\frac{2\beta}{\alpha}} \\ + 2\alpha^2\beta^6b^2e^{\frac{-2\beta}{\alpha}} - 2\beta^8e^{\frac{-2\beta}{\alpha}}) \times (-h\alpha^2\beta^3e^{\frac{2\beta}{\alpha}} + 2h\alpha^2\beta^3 \\ + 2h\beta^5e^{\frac{-2\beta}{\alpha}} + 6s_0\beta^4e^{\frac{-2\beta}{\alpha}})](b^2\alpha^2 - \beta^2) = 0. \end{array} \right. \quad (3.1.43)$$

The terms of (3.1.43) that do not contain α^2 are $(4s_0 + 2\beta h)\beta^{14}$. Hence, there exists $hp(13)v_{13}$ such that $(4s_0 + 2\beta h)\beta^{14} = \alpha^2v_{13}$. Since $\alpha \not\equiv 0 \pmod{\beta}$, we must have $v_{13} = 0$.

Thus, we have

$$(4s_0 + 2\beta h)\beta^{14} = 0, \quad (3.1.44)$$

which implies $4s_i + 2b_ih = 0$. Transvecting this equation by b^i , we obtain $hb^2 = 0$.

If $b^2 = 0$, we get from (3.1.34) and (3.1.44)

$$\left\{ \begin{array}{l} -h/2[(2\alpha^4e^{\frac{2\beta}{\alpha}} + 8\alpha^2\beta^2 + 6\beta^4e^{\frac{-2\beta}{\alpha}}) \times (-2\alpha^4e^{\frac{2\beta}{\alpha}} - 6\alpha^2\beta^2 \\ + 2\alpha^2\beta^2e^{\frac{2\beta}{\alpha}} - 2\alpha^4 - 2\alpha^2\beta^2e^{\frac{-2\beta}{\alpha}} - 4\alpha^2\beta^2e^{\frac{-2\beta}{\alpha}} \\ - 2\beta^4e^{\frac{-2\beta}{\alpha}}) - (-4\alpha^4\beta e^{\frac{2\beta}{\alpha}} - 10\alpha^2\beta^3 + 2\alpha^2\beta^3e^{\frac{2\beta}{\alpha}} \\ - 2\beta^5e^{\frac{-2\beta}{\alpha}}) \times (2\alpha^2\beta e^{\frac{2\beta}{\alpha}} + 4\alpha^2\beta + 8\beta^3e^{\frac{-2\beta}{\alpha}})] \\ - [(2\alpha^4e^{\frac{2\beta}{\alpha}} + 8\alpha^2\beta^2 + 6\beta^4e^{\frac{-2\beta}{\alpha}}) \times \\ (h\alpha^2\beta^2e^{\frac{2\beta}{\alpha}} - 3h\alpha^2\beta^2 - h\beta^4e^{\frac{-2\beta}{\alpha}} - h\alpha^2\beta^2e^{\frac{2\beta}{\alpha}} \\ + h\alpha^2\beta^2 + 2h\alpha^2\beta^2e^{\frac{-2\beta}{\alpha}} + h\beta^4e^{\frac{-2\beta}{\alpha}}) \end{array} \right. \quad (3.1.45)$$

$$\begin{cases} -((-4\alpha^4\beta^2e^{\frac{2\beta}{\alpha}} - 10\alpha^2\beta^4 + 2\alpha^2\beta^4e^{\frac{2\beta}{\alpha}} \\ -2\beta^6e^{\frac{-2\beta}{\alpha}}) \times (-h\alpha^2e^{\frac{2\beta}{\alpha}} + 2h\alpha^2 \\ +2h\beta^2e^{\frac{-2\beta}{\alpha}} - 3h\beta^2e^{\frac{-2\beta}{\alpha}})] = 0. \end{cases}$$

The term in (3.1.45) that does not include β is $4h\alpha^8$. Thus, there exists $hp(7)v_7$ such that $4h\alpha^8 = \beta v_7$. Since $\alpha^2 \not\equiv 0 \pmod{\beta}$, we have $v_7 = 0$, which leads to a contradiction. Thus we have $h=0$ and hence $s_0 = 0, r_{00} = 0$ from (3.1.44) and (3.1.42) respectively. Therefore (3.1.32) is reduced to $A(s_0^i y^j - s_0^j y^i) = 0$.

Since $A \neq 0$, we have $s_0^i y^j - s_0^j y^i = 0$. Transvection of this equation by y_j gives $s_0^i = 0$. Finally $r_{ij} = s_{ij} = 0$ are concluded, that is, $b_{i,j}=0$.

Case 3: If $v_{15} \neq 0$ and $\alpha^2 \equiv 0 \pmod{\beta}$, then Lemma (3.1.2) shows that $n = 2, b^2 = 0$ and $\alpha^2 = \beta\delta$, where $\delta = d_i(x)y^i$. Therefore, (3.1.32) gives

$$A'(s_0^i y^j - s_0^j y^i) + B'(b^i y^j - b^j y^i) = 0, \quad (3.1.46)$$

where,

$$\begin{cases} A' = [(2\delta^2 e^{\frac{2\beta}{\alpha}} + 8\beta\delta + 6\beta^2 e^{\frac{-2\beta}{\alpha}}) \times (-2\delta^2 e^{\frac{2\beta}{\alpha}} - 6\beta\delta + 2\beta\delta e^{\frac{2\beta}{\alpha}} - 2\delta^2 \\ -2\delta\beta e^{\frac{-2\beta}{\alpha}} - 4\delta\beta e^{\frac{-2\beta}{\alpha}} - 2\beta^2 e^{\frac{-2\beta}{\alpha}}) - (-4\beta\delta^2 e^{\frac{2\beta}{\alpha}} - 10\beta^2\delta + 2\beta^2\delta e^{\frac{2\beta}{\alpha}} \\ -2\beta^3 e^{\frac{-2\beta}{\alpha}}) \times (2\delta e^{\frac{2\beta}{\alpha}} + 4\delta + 8\beta e^{\frac{-2\beta}{\alpha}})], \\ B' = r_{00}[(2\delta^2 e^{\frac{2\beta}{\alpha}} + 8\beta\delta + 6\beta^2 e^{\frac{-2\beta}{\alpha}}) \times (\delta e^{\frac{2\beta}{\alpha}} - 3\delta - \beta e^{\frac{-2\beta}{\alpha}}) \\ -((-4\delta^2 e^{\frac{2\beta}{\alpha}} - 10\beta\delta + 2\beta\delta e^{\frac{2\beta}{\alpha}} - 2\beta^2 e^{\frac{-2\beta}{\alpha}}) \times (-\delta e^{\frac{2\beta}{\alpha}} + 2\delta + 2\beta e^{\frac{-2\beta}{\alpha}})] \\ +s_0[(2\delta^2 e^{\frac{2\beta}{\alpha}} + 8\beta\delta + 6\beta^2 e^{\frac{-2\beta}{\alpha}}) \times (2\delta^2 e^{\frac{2\beta}{\alpha}} - 2\delta^2 - 4\delta^2 e^{\frac{-2\beta}{\alpha}} - 2\beta\delta e^{\frac{-2\beta}{\alpha}}) \\ - (6\delta e^{\frac{-2\beta}{\alpha}}) \times (-4\delta^2\beta e^{\frac{2\beta}{\alpha}} - 10\beta^2\delta + 2\beta^2\delta e^{\frac{2\beta}{\alpha}} - 2\beta^3 e^{\frac{-2\beta}{\alpha}})]. \end{cases} \quad (3.1.47)$$

Transvection of (3.1.46) by $b_i y_j$ leads to

$$\begin{cases} s_0\delta[(2\delta^2 e^{\frac{2\beta}{\alpha}} + 8\beta\delta + 6\beta^2 e^{\frac{-2\beta}{\alpha}}) \times (-2\delta^2 e^{\frac{2\beta}{\alpha}} - 6\beta\delta + 2\beta\delta e^{\frac{2\beta}{\alpha}} \\ -2\delta^2 - 2\delta\beta e^{\frac{-2\beta}{\alpha}} - 4\delta\beta e^{\frac{-2\beta}{\alpha}} - 2\beta^2 e^{\frac{-2\beta}{\alpha}}) - (-4\beta\delta^2 e^{\frac{2\beta}{\alpha}} \\ -10\beta^2\delta + 2\beta^2\delta e^{\frac{2\beta}{\alpha}} - 2\beta^3 e^{\frac{-2\beta}{\alpha}}) \times (2\delta e^{\frac{2\beta}{\alpha}} + 4\delta \\ +8\beta e^{\frac{-2\beta}{\alpha}})] \end{cases} \quad (3.1.48)$$

$$\left\{ \begin{array}{l} -\beta r_{00}[(2\delta^2 e^{\frac{2\beta}{\alpha}} + 8\beta\delta + 6\beta^2 e^{\frac{-2\beta}{\alpha}}) \\ \times (\delta e^{\frac{2\beta}{\alpha}} - 3\delta - \beta e^{\frac{-2\beta}{\alpha}}) - ((-4\delta^2 e^{\frac{2\beta}{\alpha}} - 10\beta\delta + 2\beta\delta e^{\frac{2\beta}{\alpha}} \\ - 2\beta^2 e^{\frac{-2\beta}{\alpha}}) \times (-\delta e^{\frac{2\beta}{\alpha}} + 2\delta + 2\beta e^{\frac{-2\beta}{\alpha}})] + s_0[(2\delta^2 e^{\frac{2\beta}{\alpha}} + 8\beta\delta \\ + 6\beta^2 e^{\frac{-2\beta}{\alpha}}) \times (2\delta^2 e^{\frac{2\beta}{\alpha}} - 2\delta^2 - 4\delta^2 e^{\frac{-2\beta}{\alpha}} - 2\beta\delta e^{\frac{-2\beta}{\alpha}}) \\ - (6\delta e^{\frac{-2\beta}{\alpha}}) \times (-4\delta^2 \beta e^{\frac{2\beta}{\alpha}} - 10\beta^2 \delta + 2\beta^2 \delta e^{\frac{2\beta}{\alpha}} - 2\beta^3 e^{\frac{-2\beta}{\alpha}})] = 0. \end{array} \right.$$

The term of (3.1.48) that does not contain β is $-8\delta^5 s_0$. Thus, there exists $hp(5) v_5$ such that $-8s_0\delta^5 = \beta v_5$, which implies $s_0 = k\beta$, where $k = k(x)$.

On the other hand, (3.1.35) gives $r_{00}\beta^{14} = \delta v_{15}$, which must be reduced to,

$$r_{00} = \delta v_1. \quad (3.1.49)$$

Where v_1 is an $hp(1)$. Substituting $s_0 = k\beta$ and (3.1.49) into (3.1.48), we have

$$\left\{ \begin{array}{l} K\delta[(2\delta^2 e^{\frac{2\beta}{\alpha}} + 8\beta\delta + 6\beta^2 e^{\frac{-2\beta}{\alpha}}) \times (-2\delta^2 e^{\frac{2\beta}{\alpha}} - 6\beta\delta + 2\beta\delta e^{\frac{2\beta}{\alpha}} \\ - 2\delta^2 - 2\delta\beta e^{\frac{-2\beta}{\alpha}} - 4\delta\beta e^{\frac{-2\beta}{\alpha}} - 2\beta^2 e^{\frac{-2\beta}{\alpha}}) - (-4\beta\delta^2 e^{\frac{2\beta}{\alpha}} \\ - 10\beta^2 \delta + 2\beta^2 \delta e^{\frac{2\beta}{\alpha}} - 2\beta^3 e^{\frac{-2\beta}{\alpha}}) \times (2\delta e^{\frac{2\beta}{\alpha}} + 4\delta + 8\beta e^{\frac{-2\beta}{\alpha}})] \\ - \delta v_1[(2\delta^2 e^{\frac{2\beta}{\alpha}} + 8\beta\delta + 6\beta^2 e^{\frac{-2\beta}{\alpha}}) \times (\delta e^{\frac{2\beta}{\alpha}} - 3\delta - \beta e^{\frac{-2\beta}{\alpha}}) \\ - ((-4\delta^2 e^{\frac{2\beta}{\alpha}} - 10\beta\delta + 2\beta\delta e^{\frac{2\beta}{\alpha}} - 2\beta^2 e^{\frac{-2\beta}{\alpha}}) \times \\ (-8e^{\frac{2\beta}{\alpha}} + 2\delta + 2\beta e^{\frac{-2\beta}{\alpha}})] + k\beta[(2\delta^2 e^{\frac{2\beta}{\alpha}} + 8\beta\delta + 6\beta^2 e^{\frac{-2\beta}{\alpha}}) \\ \times (2\delta^2 e^{\frac{2\beta}{\alpha}} - 2\delta^2 - 4\delta^2 e^{\frac{-2\beta}{\alpha}} - 2\beta\delta e^{\frac{-2\beta}{\alpha}}) - (6\delta e^{\frac{-2\beta}{\alpha}}) \\ \times (-4\delta^2 \beta e^{\frac{2\beta}{\alpha}} - 10\beta^2 \delta + 2\beta^2 \delta e^{\frac{2\beta}{\alpha}} - 2\beta^3 e^{\frac{-2\beta}{\alpha}})] = 0. \end{array} \right. \quad (3.1.50)$$

The term of (3.1.50) that does not contain β is $-8\delta^5 k$. Therefore, there exists $hp(4) v_4$ such that $-8\delta^5 k = \beta v_4$, which leads to a contradiction. Thus, we have $k = 0, s_0 = 0$ and $r_{00} = 0$. From (3.1.46), we get

$$A'(s_0^i y^j - s_0^j y^i) = 0. \quad (3.1.51)$$

If

$$\begin{cases} A' = [(2\delta^2 e^{\frac{2\beta}{\alpha}} + 8\beta\delta + 6\beta^2 e^{-\frac{2\beta}{\alpha}}) \times (-2\delta^2 e^{\frac{2\beta}{\alpha}} - 6\beta\delta + 2\beta\delta e^{\frac{2\beta}{\alpha}} - 2\delta^2 \\ -2\delta\beta e^{-\frac{2\beta}{\alpha}} - 4\delta\beta e^{-\frac{2\beta}{\alpha}} - 2\beta^2 e^{-\frac{2\beta}{\alpha}}) - (-4\beta\delta^2 e^{\frac{2\beta}{\alpha}} - 10\beta^2\delta + 2\beta^2\delta e^{\frac{2\beta}{\alpha}} \\ -2\beta^3 e^{-\frac{2\beta}{\alpha}}) \times (2\delta e^{\frac{2\beta}{\alpha}} + 4\delta + 8\beta e^{-\frac{2\beta}{\alpha}})] = 0. \end{cases} \quad (3.1.52)$$

The term of (3.1.52) that does not contain β is $-8\delta^4$. Therefore, there exists $hp(3) v_3$ such that $-8\delta^4 k = \beta v_3$, which leads to a contradiction. Thus $A' \neq 0$ in (3.1.52). Hence, we get $s_0^i y^j - s_0^j y^i = 0$. Transvecting this equation by y_j gives $s_0^i = 0$. Thus, $s_{ij} = r_{ij} = 0$ is concluded, that is, $b_{i;j} = 0$.

Conversely, if $b_{i;j} = 0$, then we obtain $B^{ij} = 0$ from (3.1.6). Hence F^n is a Douglas space. Consequently, we have

Theorem 3.1.6. *An n -dimensional Finsler space F^n with an exponential (α, β) -metric $F(\alpha, \beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$ is a Douglas space if and only if*

1. $\alpha^2 \not\equiv 0 \pmod{\beta} : b_{j;i} = 0$.
2. $\alpha^2 \equiv 0 \pmod{\beta} : n = 2$ and $b_{j;i} = 0$, where $\alpha^2 = \beta\delta$, $\delta = d_i(x)y^i$ and $k = k(x)$.

Theorem 3.1.7. *If an n -dimensional Finsler space F^n with an exponential (α, β) -metric $F(\alpha, \beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$ is a Douglas space, then F^n is a Berwald space.*

3.2 The Condition to be a Weakly-Berwald Space

In this section, we have obtained the condition for a Finsler space F^n with an exponential (α, β) -metric (3.1.1) is a weakly Berwald space.

The partial derivatives with respect to α and β of (3.1.1) are given by (3.1.2).

As the function $F(\alpha, \beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$ is a positively homogeneous function of α and β of degree one, we have

$$F_\alpha \alpha + F_\beta \beta = \left(\frac{\alpha^2 e^{\frac{\beta}{\alpha}} - \alpha \beta e^{\frac{\beta}{\alpha}} + \beta^2 e^{-\frac{\beta}{\alpha}}}{\alpha^2} \right) \alpha + \left(\frac{\alpha e^{\frac{\beta}{\alpha}} + \alpha e^{-\frac{\beta}{\alpha}} - \beta e^{-\frac{\beta}{\alpha}}}{\alpha} \right) \beta = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}} = F(\alpha, \beta),$$

$$F_{\alpha\alpha}\alpha + F_{\alpha\beta}\beta = \alpha\left(\frac{\beta^2 e^{\frac{\beta}{\alpha}}}{\alpha^3} - \frac{2\beta^2 e^{\frac{-\beta}{\alpha}}}{\alpha^3} + \frac{\beta^3 e^{\frac{-\beta}{\alpha}}}{\alpha^4}\right) + \frac{\beta(-\alpha\beta e^{\frac{\beta}{\alpha}} + 2\alpha\beta e^{\frac{-\beta}{\alpha}} - \beta^2 e^{\frac{-\beta}{\alpha}})}{\alpha^3} = 0,$$

$$\begin{aligned} F_{\alpha\alpha\alpha}\alpha + F_{\alpha\alpha\beta}\beta &= \left(\frac{-3\alpha^2\beta^2 e^{\frac{\beta}{\alpha}} - \alpha\beta^3 e^{\frac{\beta}{\alpha}} + 6\alpha^2\beta^2 e^{\frac{-\beta}{\alpha}} - 6\alpha\beta^3 e^{\frac{-\beta}{\alpha}} + \beta^4 e^{\frac{-\beta}{\alpha}}}{\alpha^6}\right)\alpha + \\ &\left(\frac{2\alpha^2\beta e^{\frac{\beta}{\alpha}} + \alpha\beta^2 e^{\frac{\beta}{\alpha}} - 4\alpha^2\beta e^{\frac{-\beta}{\alpha}} + 5\alpha\beta^2 e^{\frac{-\beta}{\alpha}} - \beta^3 e^{\frac{-\beta}{\alpha}}}{\alpha^5}\right)\beta = -\left(\frac{\beta^2 e^{\frac{\beta}{\alpha}}}{\alpha^3} - \frac{2\beta^2 e^{\frac{-\beta}{\alpha}}}{\alpha^3} + \frac{\beta^3 e^{\frac{-\beta}{\alpha}}}{\alpha^4}\right) = \\ &-F_{\alpha\alpha} \end{aligned}$$

$$\alpha F_{\beta\alpha} + \beta F_{\beta\beta} = \left(\frac{-\alpha\beta e^{\frac{\beta}{\alpha}} + 2\alpha\beta e^{\frac{-\beta}{\alpha}} - \beta^2 e^{\frac{-\beta}{\alpha}}}{\alpha^3}\right)\alpha + \left(\frac{\alpha e^{\frac{\beta}{\alpha}} - 2\alpha e^{\frac{-\beta}{\alpha}} + \beta e^{\frac{-\beta}{\alpha}}}{\alpha^2}\right)\beta = 0.$$

Substituting (3.1.2) into (3.1.7), (3.1.8), (3.1.15) and (3.1.17), we have

$$C^* = \frac{\alpha^2\beta}{2} \left\{ \frac{M}{N} \right\}.$$

Where,

$$\begin{aligned} M &= r_{00}(\alpha^2 e^{\frac{\beta}{\alpha}} - \alpha\beta e^{\frac{\beta}{\alpha}} + \beta^2 e^{\frac{-\beta}{\alpha}}) - 2S_0\alpha^2(\alpha e^{\frac{\beta}{\alpha}} + \alpha e^{\frac{-\beta}{\alpha}} - \beta e^{\frac{-\beta}{\alpha}}), \\ N &= \alpha\beta^2(\alpha^2 e^{\frac{\beta}{\alpha}} - \alpha\beta e^{\frac{\beta}{\alpha}} + \beta^2 e^{\frac{-\beta}{\alpha}}) + \alpha^2 b^2(\alpha\beta^2 e^{\frac{\beta}{\alpha}} - 2\alpha\beta^2 e^{\frac{-\beta}{\alpha}} + \beta^3 e^{\frac{-\beta}{\alpha}}) \\ &- \beta^2(\alpha\beta^2 e^{\frac{\beta}{\alpha}} - 2\alpha\beta^2 e^{\frac{-\beta}{\alpha}} + \beta^3 e^{\frac{-\beta}{\alpha}}). \end{aligned}$$

$$E^* = \left(\frac{\alpha\beta e^{\frac{\beta}{\alpha}} + \alpha\beta e^{\frac{-\beta}{\alpha}} - \beta^2 e^{\frac{-\beta}{\alpha}}}{\alpha^2 e^{\frac{\beta}{\alpha}} + \alpha\beta e^{\frac{-\beta}{\alpha}}}\right) \times C^*$$

$$\left\{ \begin{aligned} B^m &= C^* \times \left\{ \left[\frac{\alpha\beta e^{\frac{\beta}{\alpha}} + \alpha\beta e^{\frac{-\beta}{\alpha}} - \beta^2 e^{\frac{-\beta}{\alpha}}}{\alpha^2 e^{\frac{\beta}{\alpha}} + \alpha\beta e^{\frac{-\beta}{\alpha}}} - \left(\frac{\alpha\beta^2 e^{\frac{\beta}{\alpha}} - 2\alpha\beta^2 e^{\frac{-\beta}{\alpha}} + \beta^3 e^{\frac{-\beta}{\alpha}}}{\alpha^4 e^{\frac{\beta}{\alpha}} - \alpha^3\beta e^{\frac{\beta}{\alpha}} + \alpha^2\beta^2 e^{\frac{-\beta}{\alpha}}} \right) \right] y^m \right. \\ &\left. + \left(\frac{\alpha\beta e^{\frac{\beta}{\alpha}} - 2\alpha\beta e^{\frac{-\beta}{\alpha}} + \beta^2 e^{\frac{-\beta}{\alpha}}}{\alpha^2 e^{\frac{\beta}{\alpha}} - \alpha\beta e^{\frac{\beta}{\alpha}} + \beta^2 e^{\frac{-\beta}{\alpha}}} \right) b^m \right\} + \left(\frac{\alpha^3 e^{\frac{\beta}{\alpha}} + \alpha^3 e^{\frac{-\beta}{\alpha}} - \alpha^2\beta e^{\frac{-\beta}{\alpha}}}{\alpha^2 e^{\frac{\beta}{\alpha}} - \alpha\beta e^{\frac{\beta}{\alpha}} + \beta^2 e^{\frac{-\beta}{\alpha}}} \right) s_0^m. \end{aligned} \right. \quad (3.2.1)$$

$$W = \frac{r_{00}(\alpha^2 e^{\frac{\beta}{\alpha}} - \alpha\beta e^{\frac{\beta}{\alpha}} + \beta^2 e^{\frac{-\beta}{\alpha}}) - 2S_0\alpha^2(\alpha e^{\frac{\beta}{\alpha}} + \alpha e^{\frac{-\beta}{\alpha}} - \beta e^{\frac{-\beta}{\alpha}})}{\alpha^2}. \quad (3.2.2)$$

$$\Omega = \frac{P}{Q}. \quad (3.2.3)$$

Where,

$$P = \alpha^3 \beta^2 e^{\frac{\beta}{\alpha}} - \alpha^2 \beta^3 e^{\frac{\beta}{\alpha}} + \alpha \beta^4 e^{-\frac{\beta}{\alpha}} + \alpha^3 \beta^2 b^2 e^{\frac{\beta}{\alpha}} - 2\alpha^3 \beta^2 e^{-\frac{\beta}{\alpha}} b^2 + \alpha^2 \beta^3 e^{-\frac{\beta}{\alpha}} b^2 - \alpha \beta^4 e^{\frac{\beta}{\alpha}} + 2\alpha \beta^4 e^{-\frac{\beta}{\alpha}} - \beta^5 e^{-\frac{\beta}{\alpha}},$$

$$Q = \alpha^3.$$

$$\left\{ \begin{aligned} A = & \left(\frac{n+1}{\alpha^5} \right) (-2\alpha^4 \beta^4 e^{\frac{3\beta}{\alpha}} + \alpha^5 \beta^3 e^{\frac{3\beta}{\alpha}} + \alpha^5 \beta^3 e^{\frac{\beta}{\alpha}} + 4\alpha^3 \beta^5 e^{-\frac{\beta}{\alpha}} \\ & - 5\alpha^2 \beta^6 e^{-\frac{\beta}{\alpha}} + 2\alpha^3 \beta^5 e^{\frac{3\beta}{\alpha}} - \alpha^4 \beta^4 e^{\frac{\beta}{\alpha}} - \alpha^4 \beta^4 e^{\frac{3\beta}{\alpha}} + 2\alpha \beta^7 e^{-\frac{\beta}{\alpha}} \\ & - 2\alpha^2 \beta^6 e^{\frac{\beta}{\alpha}} + \alpha^3 \beta^5 e^{\frac{\beta}{\alpha}} + 3\alpha \beta^7 e^{-\frac{3\beta}{\alpha}} - 2\beta^8 e^{-\frac{3\beta}{\alpha}}) \\ & + \frac{1}{\alpha^5} (4b^2 \alpha^3 \beta^5 e^{\frac{\beta}{\alpha}} - 8b^2 \alpha^4 \beta^4 e^{\frac{\beta}{\alpha}} + 8b^2 \alpha^4 \beta^4 e^{-\frac{\beta}{\alpha}} + b^2 \alpha^6 \beta^2 e^{\frac{3\beta}{\alpha}} \\ & - 2b^2 \alpha^6 \beta^2 e^{\frac{\beta}{\alpha}} + 6b^2 \alpha^5 \beta^3 e^{\frac{\beta}{\alpha}} + 4b^2 \alpha^2 \beta^6 e^{-\frac{\beta}{\alpha}} - 8b^2 \alpha^3 \beta^5 e^{-\frac{\beta}{\alpha}} \\ & + 2b^2 \alpha^3 \beta^5 e^{-\frac{3\beta}{\alpha}} + b^2 \alpha^5 \beta^3 e^{\frac{\beta}{\alpha}} - 2b^2 \alpha^5 \beta^3 e^{-\frac{\beta}{\alpha}} - 4\beta^8 e^{-\frac{\beta}{\alpha}} + 8\alpha \beta^7 e^{-\frac{\beta}{\alpha}} \\ & - 2\alpha \beta^7 e^{-\frac{3\beta}{\alpha}} - \alpha^3 \beta^5 e^{\frac{\beta}{\alpha}} + 2\alpha^3 \beta^5 e^{-\frac{\beta}{\alpha}} - 8\alpha^2 \beta^6 e^{-\frac{\beta}{\alpha}} - 4\alpha \beta^7 e^{\frac{\beta}{\alpha}} \\ & + 8\alpha^2 \beta^6 e^{\frac{\beta}{\alpha}} - \alpha^4 \beta^4 e^{\frac{3\beta}{\alpha}} + 2\alpha^4 \beta^4 e^{\frac{\beta}{\alpha}} - 6\alpha^3 \beta^5 e^{\frac{\beta}{\alpha}}), \end{aligned} \right. \quad (3.2.4)$$

$$B = \frac{\alpha^2 \beta^2 e^{\frac{2\beta}{\alpha}} - 2\alpha^2 \beta^2 + 2\alpha \beta^3 - 2\alpha \beta^3 e^{-\frac{2\beta}{\alpha}} + \beta^4 e^{-\frac{2\beta}{\alpha}}}{\alpha^2}, \quad (3.2.5)$$

$$\left\{ \begin{aligned} C = & \frac{\beta(b^2 \alpha^2 - \beta^2)}{\alpha^5} \times (-\alpha^5 \beta^2 e^{\frac{2\beta}{\alpha}} - 8\alpha^3 \beta^4 - 3\alpha \beta^6 e^{-\frac{2\beta}{\alpha}} + 2\alpha^4 \beta^3 e^{\frac{2\beta}{\alpha}} \\ & + 14\alpha^2 \beta^5 + 2\alpha^3 \beta^4 e^{\frac{2\beta}{\alpha}} - b^2 \alpha^5 \beta^2 e^{\frac{2\beta}{\alpha}} + 2\alpha^5 \beta^2 b^2 - 6b^2 \alpha^4 \beta^3 \\ & - 8b^2 \alpha^3 \beta^4 e^{-\frac{2\beta}{\alpha}} + 6b^2 \alpha^2 \beta^5 e^{-\frac{2\beta}{\alpha}} - 4b^2 \alpha^2 \beta^5 - b^2 \alpha \beta^6 e^{-\frac{2\beta}{\alpha}} \\ & + 8b^2 \alpha^3 \beta^4 + 4\beta^7 - 8\alpha \beta^6 + 2\beta^7 e^{-\frac{2\beta}{\alpha}} - 2\alpha^2 \beta^5 e^{\frac{2\beta}{\alpha}}), \end{aligned} \right. \quad (3.2.6)$$

$$\left\{ \begin{aligned} D = & \frac{1}{\alpha^3} \times (-4b^2 \alpha^5 \beta^4 e^{\frac{2\beta}{\alpha}} + 10b^2 \alpha^4 \beta^5 + 2b^2 \alpha^6 \beta^3 e^{\frac{2\beta}{\alpha}} + 2b^2 \alpha^6 \beta^3 \\ & + 26b^2 \alpha^4 \beta^5 e^{-\frac{2\beta}{\alpha}} - 30b^2 \alpha^3 \beta^6 e^{-\frac{2\beta}{\alpha}} + 6\alpha^3 \beta^6 e^{\frac{2\beta}{\alpha}} + 4\alpha^3 \beta^6 - 12\alpha^2 \beta^7 \\ & - 4\alpha^4 \beta^5 e^{\frac{2\beta}{\alpha}} - 4\alpha^4 \beta^5 - 28\alpha^2 \beta^7 e^{-\frac{2\beta}{\alpha}} + 4\alpha \beta^8 e^{-\frac{2\beta}{\alpha}} - 18b^2 \alpha^4 \beta^5 e^{\frac{2\beta}{\alpha}} \\ & + 10b^2 \alpha^2 \beta^7 e^{-\frac{2\beta}{\alpha}} + 8\alpha^2 \beta^7 e^{\frac{2\beta}{\alpha}} + 2b^4 \alpha^6 \beta^3 e^{\frac{2\beta}{\alpha}} - 2b^4 \alpha^6 \beta^3 + 8b^4 \alpha^5 \beta^4 \end{aligned} \right. \quad (3.2.7)$$

$$\left\{ \begin{array}{l} -4b^4\alpha^6\beta^3e^{\frac{-2\beta}{\alpha}} + 20b^4\alpha^5\beta^4e^{\frac{-2\beta}{\alpha}} + 4b^4\alpha^4\beta^5e^{\frac{-2\beta}{\alpha}} + 8b^2\alpha^4\beta^5e^{\frac{2\beta}{\alpha}} \\ -32b^2\alpha^3\beta^6 + 24\alpha\beta^8 - 8b^4\alpha^4\beta^5 + 16b^2\alpha^2\beta^7 - 8\beta^9 - 14b^4\alpha^4\beta^5e^{\frac{-2\beta}{\alpha}} \end{array} \right\},$$

$$\left\{ \begin{array}{l} E = \frac{1}{\alpha^3} \times (2\alpha^5\beta^4e^{\frac{2\beta}{\alpha}} - 4\alpha^4\beta^5e^{\frac{2\beta}{\alpha}} + 8\alpha^3\beta^6 + 2b^2\alpha^5\beta^4e^{\frac{2\beta}{\alpha}} \\ -4\alpha^5\beta^4b^2 + 6b^2\alpha^4\beta^5 - 10\alpha^2\beta^7 - 2b^2\alpha^4\beta^5e^{\frac{2\beta}{\alpha}} + 2\alpha^2\beta^7e^{\frac{2\beta}{\alpha}} \\ + 6\alpha\beta^8e^{\frac{-2\beta}{\alpha}} - 4b^2\alpha^3\beta^6e^{\frac{-2\beta}{\alpha}} + 2b^2\alpha^2\beta^7e^{\frac{-2\beta}{\alpha}} - 2\beta^9e^{\frac{-2\beta}{\alpha}}). \end{array} \right. \quad (3.2.8)$$

Substituting (3.2.4), (3.2.5), (3.2.6), (3.2.7), (3.2.8) into (3.1.16), we have

$$\left\{ \begin{array}{l} B^{mm}\{a_0\alpha^{11}\beta^2 + a_1\alpha^{10}\beta^3 + a_2\alpha^9\beta^4 + a_3\alpha^8\beta^5 + a_4\alpha^7\beta^6 \\ + a_5\alpha^6\beta^7 + a_6\alpha^5\beta^8 + a_7\alpha^4\beta^9 + a_8\alpha^3\beta^{10} + a_9\alpha^2\beta^{11} \\ + a_{10}\alpha\beta^{12} + a_{11}\beta^{13}\} + r_{00}\{a_{12}\alpha^{10}\beta^2 + a_{13}\alpha^9\beta^3 \\ + a_{14}\alpha^8\beta^4 + a_{15}\alpha^7\beta^5 + a_{16}\alpha^6\beta^6 + a_{17}\alpha^5\beta^7 \\ + a_{18}\alpha^4\beta^8 + a_{19}\alpha^3\beta^9 + a_{20}\alpha^2\beta^{10} + a_{21}\alpha\beta^{11} \\ + a_{22}\beta^{12}\} + s_0\{a_{23}\alpha^{13} + a_{24}\alpha^{12}\beta + a_{25}\alpha^{11}\beta^2 \\ + a_{26}\alpha^{10}\beta^3 + a_{27}\alpha^9\beta^4 + a_{28}\alpha^8\beta^5 + a_{29}\alpha^7\beta^6 \\ + a_{30}\alpha^6\beta^7 + a_{31}\alpha^5\beta^8 + a_{32}\alpha^4\beta^9 + a_{33}\alpha^3\beta^{10} \\ + a_{34}\alpha^2\beta^{11}\} + r_0\{a_{35}\alpha^{11}\beta^2 + a_{36}\alpha^{10}\beta^3 \\ + a_{37}\alpha^9\beta^4 + a_{38}\alpha^8\beta^5 + a_{39}\alpha^7\beta^6 + a_{40}\alpha^6\beta^7 \\ + a_{41}\alpha^5\beta^8 + a_{42}\alpha^4\beta^9 + a_{43}\alpha^3\beta^{10} + a_{44}\alpha^2\beta^{11}\}. \end{array} \right. \quad (3.2.9)$$

Where,

$$\begin{aligned} a_0 &= (2e^{\frac{5\beta}{\alpha}} + 4b^2e^{\frac{5\beta}{\alpha}} - 8b^2e^{\frac{3\beta}{\alpha}} + 2b^4e^{\frac{5\beta}{\alpha}} - 8b^4e^{\frac{3\beta}{\alpha}} + 8b^4e^{\frac{\beta}{\alpha}}), \\ a_1 &= (-8e^{\frac{5\beta}{\alpha}} + 32b^2e^{\frac{3\beta}{\alpha}} - 12b^2e^{\frac{5\beta}{\alpha}} + 22b^4e^{\frac{3\beta}{\alpha}} - 32b^4e^{\frac{\beta}{\alpha}} - 4b^4e^{\frac{5\beta}{\alpha}} + 2e^{\frac{3\beta}{\alpha}} - 8b^2e^{\frac{\beta}{\alpha}} + 8b^4e^{\frac{-\beta}{\alpha}}), \\ a_2 &= (8e^{\frac{5\beta}{\alpha}} + 8e^{\frac{3\beta}{\alpha}} - 20b^2e^{\frac{3\beta}{\alpha}} + 8b^2e^{\frac{5\beta}{\alpha}} + 30b^4e^{\frac{\beta}{\alpha}} - 16b^4e^{\frac{3\beta}{\alpha}} + 2b^4e^{\frac{5\beta}{\alpha}} - 8b^4e^{\frac{-\beta}{\alpha}} - 12b^2e^{\frac{\beta}{\alpha}}), \\ a_3 &= (-44e^{\frac{3\beta}{\alpha}} + 4e^{\frac{5\beta}{\alpha}} + 100b^2e^{\frac{\beta}{\alpha}} - 36b^2e^{\frac{3\beta}{\alpha}} + 4b^2e^{\frac{5\beta}{\alpha}} + 16e^{\frac{\beta}{\alpha}} - 40b^2e^{\frac{-\beta}{\alpha}} + 16b^4e^{\frac{-3\beta}{\alpha}} + 2b^4e^{\frac{3\beta}{\alpha}} - \\ & 22b^4e^{\frac{-\beta}{\alpha}}), \\ a_4 &= (44e^{\frac{3\beta}{\alpha}} - 8e^{\frac{\beta}{\alpha}} - 100b^2e^{\frac{\beta}{\alpha}} - 8e^{\frac{5\beta}{\alpha}} + 52b^2e^{\frac{-\beta}{\alpha}} - 4b^4e^{\frac{\beta}{\alpha}} + 24b^4e^{\frac{-\beta}{\alpha}} - 24b^4e^{\frac{-3\beta}{\alpha}} + 40b^2e^{\frac{3\beta}{\alpha}} - \\ & 4b^2e^{\frac{5\beta}{\alpha}} - 4b^2e^{\frac{3\beta}{\alpha}}), \end{aligned}$$

$$\begin{aligned}
a_5 &= (-68e^{\frac{\beta}{\alpha}} + 8e^{\frac{3\beta}{\alpha}} + 44b^2e^{\frac{-\beta}{\alpha}} + 8b^2e^{\frac{\beta}{\alpha}} - 4b^4e^{\frac{-\beta}{\alpha}} + 4b^4e^{\frac{-3\beta}{\alpha}} + 44e^{\frac{-\beta}{\alpha}} - 56b^2e^{\frac{-3\beta}{\alpha}} + 8b^4e^{\frac{-5\beta}{\alpha}} - 4b^2e^{\frac{3\beta}{\alpha}}), \\
a_6 &= (74e^{\frac{\beta}{\alpha}} - 60e^{\frac{-\beta}{\alpha}} - 72b^2e^{\frac{-\beta}{\alpha}} + 76b^2e^{\frac{-3\beta}{\alpha}} + 8b^2e^{\frac{\beta}{\alpha}} + 2b^4e^{\frac{-3\beta}{\alpha}} - 20e^{\frac{3\beta}{\alpha}} + 2e^{\frac{5\beta}{\alpha}} - 8b^4e^{\frac{-5\beta}{\alpha}}), \\
a_7 &= (-18e^{\frac{-\beta}{\alpha}} - 8e^{\frac{\beta}{\alpha}} - 12b^2e^{\frac{-3\beta}{\alpha}} + 12b^2e^{\frac{-\beta}{\alpha}} + 48e^{\frac{-3\beta}{\alpha}} - 23b^2e^{\frac{-5\beta}{\alpha}} - 4b^2e^{\frac{-\beta}{\alpha}} + 2b^4e^{\frac{-5\beta}{\alpha}} + 2e^{\frac{3\beta}{\alpha}}), \\
a_8 &= (40e^{\frac{-\beta}{\alpha}} - 58e^{\frac{-3\beta}{\alpha}} - 4b^2e^{\frac{-3\beta}{\alpha}} - 4e^{\frac{\beta}{\alpha}} + 20b^2e^{\frac{-5\beta}{\alpha}}), \\
a_9 &= (8e^{\frac{-3\beta}{\alpha}} - 4e^{\frac{-\beta}{\alpha}} - 4b^2e^{\frac{-5\beta}{\alpha}}), \\
a_{10} &= (2e^{\frac{-3\beta}{\alpha}} - 12e^{\frac{-5\beta}{\alpha}}), \\
a_{11} &= (2e^{\frac{-5\beta}{\alpha}}), \\
a_{12} &= (n(-e^{\frac{5\beta}{\alpha}} - e^{\frac{3\beta}{\alpha}} - b^2e^{\frac{5\beta}{\alpha}} + b^2e^{\frac{3\beta}{\alpha}} + 2b^2) - e^{\frac{5\beta}{\alpha}} - e^{\frac{3\beta}{\alpha}} - 2b^2e^{\frac{5\beta}{\alpha}} - 6b^2e^{\frac{3\beta}{\alpha}} + 2b^2 + 12b^4e^{\frac{3\beta}{\alpha}}), \\
a_{13} &= (n(5e^{\frac{5\beta}{\alpha}} + 3e^{\frac{3\beta}{\alpha}} - 5b^2e^{\frac{\beta}{\alpha}} + 4b^2e^{\frac{5\beta}{\alpha}} - 7b^2e^{\frac{3\beta}{\alpha}}) + 5e^{\frac{5\beta}{\alpha}} + 3e^{\frac{3\beta}{\alpha}} + 2b^2e^{\frac{\beta}{\alpha}} + 4b^2e^{\frac{5\beta}{\alpha}} + 8b^2e^{\frac{3\beta}{\alpha}} + 32b^4e^{\frac{3\beta}{\alpha}} + 48b^4e^{\frac{-\beta}{\alpha}} - 45b^4e^{\frac{\beta}{\alpha}} - 51b^4e^{\frac{\beta}{\alpha}}), \\
a_{14} &= n(-8e^{\frac{\beta}{\alpha}} - 7e^{\frac{3\beta}{\alpha}} - 8e^{\frac{5\beta}{\alpha}} + 10b^2e^{\frac{-\beta}{\alpha}} + 11b^2e^{\frac{3\beta}{\alpha}} + 3b^2e^{\frac{\beta}{\alpha}} - 5b^2e^{\frac{5\beta}{\alpha}}) - 6e^{\frac{\beta}{\alpha}} - 7e^{\frac{5\beta}{\alpha}} - 2b^2e^{\frac{-\beta}{\alpha}} - 13b^2e^{\frac{3\beta}{\alpha}} + 66b^2e^{\frac{\beta}{\alpha}} + 2b^2e^{\frac{5\beta}{\alpha}} - 35b^2e^{\frac{\beta}{\alpha}} + 47b^4e^{\frac{\beta}{\alpha}} + 36b^4e^{\frac{-\beta}{\alpha}} - 12b^4e^{\frac{-3\beta}{\alpha}}), \\
a_{15} &= (n(20e^{\frac{3\beta}{\alpha}} + 22e^{\frac{\beta}{\alpha}} + 3e^{\frac{5\beta}{\alpha}} - 25b^2e^{\frac{-\beta}{\alpha}} - 3b^2e^{\frac{3\beta}{\alpha}} - 5b^2e^{\frac{\beta}{\alpha}} + 2b^2e^{\frac{5\beta}{\alpha}}) - e^{\frac{3\beta}{\alpha}} + 35e^{\frac{\beta}{\alpha}} + 3e^{\frac{5\beta}{\alpha}} - 80b^2e^{\frac{-\beta}{\alpha}} - 33b^2e^{\frac{3\beta}{\alpha}} + 119b^2e^{\frac{\beta}{\alpha}} + 4b^2e^{\frac{5\beta}{\alpha}} - 24b^4e^{\frac{\beta}{\alpha}} + 11b^4e^{\frac{-\beta}{\alpha}} + 6b^4e^{\frac{-3\beta}{\alpha}} + 8b^4e^{\frac{3\beta}{\alpha}} + b^4e^{\frac{\beta}{\alpha}} + 43b^4e^{\frac{-\beta}{\alpha}}), \\
a_{16} &= (n(-26e^{\frac{3\beta}{\alpha}} + 14b^2e^{\frac{-3\beta}{\alpha}} - 24e^{\frac{\beta}{\alpha}} + 19b^2e^{\frac{-\beta}{\alpha}} - 22e^{\frac{-\beta}{\alpha}} + 3e^{\frac{5\beta}{\alpha}} + 6b^2e^{\frac{\beta}{\alpha}} - 2b^2e^{\frac{3\beta}{\alpha}}) + 19e^{\frac{3\beta}{\alpha}} + 2b^2e^{\frac{-3\beta}{\alpha}} - 67e^{\frac{\beta}{\alpha}} + 29b^2e^{\frac{-\beta}{\alpha}} - 8e^{\frac{-\beta}{\alpha}} + 11b^2e^{\frac{\beta}{\alpha}} - 8b^2e^{\frac{3\beta}{\alpha}} + 4b^4e^{\frac{-\beta}{\alpha}} - b^4e^{\frac{\beta}{\alpha}} + 4b^4e^{\frac{-3\beta}{\alpha}} - 12b^4e^{\frac{-5\beta}{\alpha}}), \\
a_{17} &= (n(52e^{\frac{-\beta}{\alpha}} + 23e^{\frac{\beta}{\alpha}} - 27b^2e^{\frac{-3\beta}{\alpha}} + 11e^{\frac{3\beta}{\alpha}} - 6b^2e^{\frac{-\beta}{\alpha}} - 2e^{\frac{5\beta}{\alpha}} - 2e^{\frac{3\beta}{\alpha}}) + 49e^{\frac{-\beta}{\alpha}} + 56e^{\frac{\beta}{\alpha}} - 18b^2e^{\frac{-3\beta}{\alpha}} - 24e^{\frac{3\beta}{\alpha}} - 8b^2e^{\frac{-\beta}{\alpha}} + 33b^2e^{\frac{\beta}{\alpha}} + b^4e^{\frac{-\beta}{\alpha}} - 5b^4e^{\frac{-3\beta}{\alpha}} + 18b^4e^{\frac{-5\beta}{\alpha}}), \\
a_{18} &= (n(-41e^{\frac{-\beta}{\alpha}} + 15b^2e^{\frac{-3\beta}{\alpha}} - 14e^{\frac{\beta}{\alpha}} - 24e^{\frac{-3\beta}{\alpha}} + 6b^2e^{\frac{-5\beta}{\alpha}} + 2e^{\frac{3\beta}{\alpha}}) - 80e^{\frac{-\beta}{\alpha}} + 20b^2e^{\frac{-3\beta}{\alpha}} - 8e^{\frac{\beta}{\alpha}} + 6b^2e^{\frac{-5\beta}{\alpha}} + 8e^{\frac{3\beta}{\alpha}} - 2b^2e^{\frac{-\beta}{\alpha}} + b^2e^{\frac{\beta}{\alpha}} + b^4e^{\frac{-3\beta}{\alpha}} - 8b^4e^{\frac{-5\beta}{\alpha}}), \\
a_{19} &= (n(42e^{\frac{-3\beta}{\alpha}} + 14e^{\frac{-\beta}{\alpha}} - 2b^2e^{\frac{-3\beta}{\alpha}} - 7b^2e^{\frac{-5\beta}{\alpha}}) + 15e^{\frac{-3\beta}{\alpha}} + 48e^{\frac{-\beta}{\alpha}} - 3b^2e^{\frac{-3\beta}{\alpha}} - 16b^2e^{\frac{-5\beta}{\alpha}} - 10e^{\frac{\beta}{\alpha}} + b^2e^{\frac{-\beta}{\alpha}} + b^4e^{\frac{-5\beta}{\alpha}}), \\
a_{20} &= (n(-21e^{\frac{-3\beta}{\alpha}} - 9e^{\frac{-5\beta}{\alpha}} + 2b^2e^{\frac{-5\beta}{\alpha}}) - 24e^{\frac{-3\beta}{\alpha}} + 3e^{\frac{-5\beta}{\alpha}} + 8b^2e^{\frac{-5\beta}{\alpha}} - 8e^{\frac{-\beta}{\alpha}} + 2e^{\frac{\beta}{\alpha}} + b^2e^{\frac{-3\beta}{\alpha}}), \\
a_{21} &= (n(5e^{\frac{-5\beta}{\alpha}} + 6e^{\frac{-3\beta}{\alpha}}) + 4e^{\frac{-5\beta}{\alpha}} + 12e^{\frac{-3\beta}{\alpha}} - 2e^{\frac{-\beta}{\alpha}} - b^2e^{\frac{-5\beta}{\alpha}}), \\
a_{22} &= (2ne^{\frac{-5\beta}{\alpha}}), \\
a_{23} &= (-2b^2e^{\frac{3\beta}{\alpha}} - 4b^2e^{\frac{\beta}{\alpha}} + 2b^2e^{\frac{5\beta}{\alpha}} + 2b^4e^{\frac{5\beta}{\alpha}} - 6b^4e^{\frac{3\beta}{\alpha}} + 8b^4e^{\frac{-\beta}{\alpha}}),
\end{aligned}$$

$$\begin{aligned}
a_{24} &= (n(2e^{\frac{5\beta}{\alpha}} + 4e^{\frac{3\beta}{\alpha}} + 2b^2e^{\frac{5\beta}{\alpha}} - 6b^2e^{\frac{\beta}{\alpha}} + 2e^{\frac{\beta}{\alpha}}) + 2e^{\frac{5\beta}{\alpha}} + 4e^{\frac{3\beta}{\alpha}} + 16b^2e^{\frac{\beta}{\alpha}} + 2e^{\frac{\beta}{\alpha}} + 16b^2e^{\frac{3\beta}{\alpha}} + 20b^4e^{\frac{3\beta}{\alpha}} + 8b^4e^{\frac{-3\beta}{\alpha}} + 12b^4e^{\frac{\beta}{\alpha}} - 4b^2e^{\frac{-\beta}{\alpha}} - 2b^2e^{\frac{5\beta}{\alpha}} - 2b^4e^{\frac{5\beta}{\alpha}} - 8b^4e^{\frac{-\beta}{\alpha}}), \\
a_{25} &= (n(-8e^{\frac{5\beta}{\alpha}} - 14e^{\frac{3\beta}{\alpha}} - 6b^2e^{\frac{5\beta}{\alpha}} + 19b^2e^{\frac{\beta}{\alpha}} + 4b^2e^{\frac{3\beta}{\alpha}} - 10e^{\frac{\beta}{\alpha}} + 10b^2e^{\frac{-\beta}{\alpha}}) - 12e^{\frac{5\beta}{\alpha}} - 8e^{\frac{3\beta}{\alpha}} - 8b^2e^{\frac{5\beta}{\alpha}} - 63b^2e^{\frac{\beta}{\alpha}} - 8b^2e^{\frac{3\beta}{\alpha}} - 10e^{\frac{\beta}{\alpha}} + 18b^2e^{\frac{-\beta}{\alpha}} + 2b^4e^{\frac{\beta}{\alpha}} + 114b^4e^{\frac{-\beta}{\alpha}} - 52b^4e^{\frac{-3\beta}{\alpha}} - 22b^4e^{\frac{3\beta}{\alpha}}), \\
a_{26} &= (n(26e^{\frac{\beta}{\alpha}} + 8e^{\frac{5\beta}{\alpha}} + 24e^{\frac{3\beta}{\alpha}} - 20b^2e^{\frac{-\beta}{\alpha}} - 22b^2e^{\frac{\beta}{\alpha}} - 2b^2e^{\frac{3\beta}{\alpha}} + 4b^2e^{\frac{5\beta}{\alpha}} + 14e^{\frac{-\beta}{\alpha}} + 116b^2e^{\frac{-3\beta}{\alpha}}) + 4e^{\frac{\beta}{\alpha}} + 18e^{\frac{5\beta}{\alpha}} + 20e^{\frac{3\beta}{\alpha}} + 46b^2e^{\frac{-\beta}{\alpha}} + 86b^2e^{\frac{\beta}{\alpha}} - 48b^2e^{\frac{3\beta}{\alpha}} + 14b^2e^{\frac{5\beta}{\alpha}} + 14e^{\frac{-\beta}{\alpha}} - 32b^2e^{\frac{-3\beta}{\alpha}} + 42b^4e^{\frac{\beta}{\alpha}} - 216b^4e^{\frac{-\beta}{\alpha}} + 156b^4e^{\frac{-3\beta}{\alpha}} + 16b^4e^{\frac{3\beta}{\alpha}} - 8b^4e^{\frac{-5\beta}{\alpha}}), \\
a_{27} &= (n(-34e^{\frac{3\beta}{\alpha}} - 58e^{\frac{\beta}{\alpha}} + 2e^{\frac{5\beta}{\alpha}} - 40e^{\frac{-\beta}{\alpha}} + 46b^2e^{\frac{-3\beta}{\alpha}} + 12b^2e^{\frac{\beta}{\alpha}} - 4b^2e^{\frac{3\beta}{\alpha}} + 26b^2e^{\frac{-\beta}{\alpha}}) + 2e^{\frac{3\beta}{\alpha}} - 18e^{\frac{\beta}{\alpha}} + 4e^{\frac{5\beta}{\alpha}} - 52e^{\frac{-\beta}{\alpha}} + 186b^2e^{\frac{-3\beta}{\alpha}} - 112b^2e^{\frac{\beta}{\alpha}} + 80b^2e^{\frac{3\beta}{\alpha}} - 168b^2e^{\frac{-\beta}{\alpha}} - 16b^4e^{\frac{\beta}{\alpha}} + 74b^4e^{\frac{-\beta}{\alpha}} - 192b^4e^{\frac{-3\beta}{\alpha}} + 20b^4e^{\frac{-5\beta}{\alpha}} - 6b^2e^{\frac{5\beta}{\alpha}} + 56b^4e^{\frac{-\beta}{\alpha}} - 8b^4e^{\frac{3\beta}{\alpha}}), \\
a_{28} &= (n(14e^{\frac{3\beta}{\alpha}} + 58e^{\frac{\beta}{\alpha}} - 12b^2e^{\frac{-\beta}{\alpha}} - 4e^{\frac{5\beta}{\alpha}} - 52b^2e^{\frac{-3\beta}{\alpha}} + 70e^{\frac{-\beta}{\alpha}} + 30e^{\frac{-3\beta}{\alpha}} - 12b^2e^{\frac{-5\beta}{\alpha}}) + 38e^{\frac{3\beta}{\alpha}} - 58e^{\frac{\beta}{\alpha}} + 232b^2e^{\frac{-\beta}{\alpha}} - 10e^{\frac{5\beta}{\alpha}} - 384b^2e^{\frac{-3\beta}{\alpha}} + 49e^{\frac{-\beta}{\alpha}} + 38e^{\frac{-3\beta}{\alpha}} + 8b^2e^{\frac{-5\beta}{\alpha}} + 128b^2e^{\frac{\beta}{\alpha}} - 40b^4e^{\frac{-\beta}{\alpha}} + 96b^4e^{\frac{-3\beta}{\alpha}} + 16b^4e^{\frac{-5\beta}{\alpha}} - 54b^2e^{\frac{3\beta}{\alpha}} - 8b^4e^{\frac{\beta}{\alpha}}), \\
a_{29} &= (n(-70e^{\frac{-\beta}{\alpha}} - 28e^{\frac{\beta}{\alpha}} + 26b^2e^{\frac{-3\beta}{\alpha}} + 26b^2e^{\frac{-5\beta}{\alpha}} + 4e^{\frac{3\beta}{\alpha}} - 78e^{\frac{-3\beta}{\alpha}}) - 10e^{\frac{-\beta}{\alpha}} + 144e^{\frac{\beta}{\alpha}} + 426b^2e^{\frac{-3\beta}{\alpha}} + 30b^2e^{\frac{-5\beta}{\alpha}} - 74e^{\frac{3\beta}{\alpha}} - 142e^{\frac{-3\beta}{\alpha}} - 154b^2e^{\frac{-\beta}{\alpha}} + 124e^{\frac{-\beta}{\alpha}} - 28b^2e^{\frac{-5\beta}{\alpha}} - 68b^2e^{\frac{\beta}{\alpha}} - 2e^{\frac{5\beta}{\alpha}} - 24b^4e^{\frac{-3\beta}{\alpha}} + 12b^4e^{\frac{-5\beta}{\alpha}} + 8b^4e^{\frac{-\beta}{\alpha}} - 40b^4e^{\frac{-5\beta}{\alpha}}), \\
a_{30} &= (n(28e^{\frac{-\beta}{\alpha}} + 86e^{\frac{-3\beta}{\alpha}} - 4b^2e^{\frac{-3\beta}{\alpha}} + 18e^{\frac{-5\beta}{\alpha}} - 18b^2e^{\frac{-5\beta}{\alpha}}) - 30e^{\frac{-\beta}{\alpha}} + 267e^{\frac{-3\beta}{\alpha}} - 248b^2e^{\frac{-3\beta}{\alpha}} + 6e^{\frac{-5\beta}{\alpha}} - 36b^2e^{\frac{-5\beta}{\alpha}} + 92b^2e^{\frac{-\beta}{\alpha}} - 126e^{\frac{\beta}{\alpha}} + 38e^{\frac{3\beta}{\alpha}} + 8b^4e^{\frac{-5\beta}{\alpha}} + 8b^4e^{\frac{-3\beta}{\alpha}} + 16b^2e^{\frac{\beta}{\alpha}}), \\
a_{31} &= (n(-38e^{\frac{-3\beta}{\alpha}} - 36e^{\frac{-5\beta}{\alpha}} + 4b^2e^{\frac{-5\beta}{\alpha}}) - 250e^{\frac{-3\beta}{\alpha}} - 40e^{\frac{-5\beta}{\alpha}} + 46b^2e^{\frac{-5\beta}{\alpha}} + 70b^2e^{\frac{-3\beta}{\alpha}} + 84e^{\frac{-\beta}{\alpha}} + 30e^{\frac{\beta}{\alpha}} - 8e^{\frac{3\beta}{\alpha}} - 16b^2e^{\frac{-\beta}{\alpha}}), \\
a_{32} &= (n(4e^{\frac{-3\beta}{\alpha}} + 22e^{\frac{-5\beta}{\alpha}}) + 168e^{\frac{-3\beta}{\alpha}} + 12e^{\frac{-5\beta}{\alpha}} - 56e^{\frac{-\beta}{\alpha}} - 6b^2e^{\frac{-5\beta}{\alpha}} - 8e^{\frac{\beta}{\alpha}} - 16b^2e^{\frac{-3\beta}{\alpha}}), \\
a_{33} &= (n(-4e^{\frac{-5\beta}{\alpha}}) + 8e^{\frac{-5\beta}{\alpha}} - 58e^{\frac{-3\beta}{\alpha}} + 8e^{\frac{-\beta}{\alpha}}), \\
a_{34} &= (-2e^{\frac{-5\beta}{\alpha}} + 8e^{\frac{-3\beta}{\alpha}}), \\
a_{35} &= (-2e^{\frac{5\beta}{\alpha}} - 2b^2e^{\frac{5\beta}{\alpha}} + 8b^2e^{\frac{3\beta}{\alpha}} + 4e^{\frac{3\beta}{\alpha}} - 8b^2e^{\frac{\beta}{\alpha}}), \\
a_{36} &= (6e^{\frac{5\beta}{\alpha}} - 22b^2e^{\frac{3\beta}{\alpha}} + 4b^2e^{\frac{5\beta}{\alpha}} - 16e^{\frac{3\beta}{\alpha}} + 32b^2e^{\frac{\beta}{\alpha}} + 4e^{\frac{\beta}{\alpha}} - 8b^2e^{\frac{-\beta}{\alpha}}), \\
a_{37} &= (10e^{\frac{3\beta}{\alpha}} + 6e^{\frac{\beta}{\alpha}} + 8b^2e^{\frac{-\beta}{\alpha}} - 30b^2e^{\frac{\beta}{\alpha}} + 16b^2e^{\frac{3\beta}{\alpha}} - 4e^{\frac{5\beta}{\alpha}} - 2b^2e^{\frac{5\beta}{\alpha}}), \\
a_{38} &= (-2e^{\frac{5\beta}{\alpha}} - 50e^{\frac{\beta}{\alpha}} + 22b^2e^{\frac{-\beta}{\alpha}} + 18e^{\frac{3\beta}{\alpha}} + 20e^{\frac{-\beta}{\alpha}} - 16b^2e^{\frac{-3\beta}{\alpha}} - 2b^2e^{\frac{3\beta}{\alpha}}), \\
a_{39} &= (50e^{\frac{\beta}{\alpha}} - 26e^{\frac{-\beta}{\alpha}} - 18e^{\frac{3\beta}{\alpha}} - 24b^2e^{\frac{-\beta}{\alpha}} + 2e^{\frac{5\beta}{\alpha}} + 4b^2e^{\frac{\beta}{\alpha}} + 24b^2e^{\frac{-3\beta}{\alpha}}), \\
a_{40} &= (-4e^{\frac{\beta}{\alpha}} - 22e^{\frac{-\beta}{\alpha}} - 4b^2e^{\frac{-3\beta}{\alpha}} + 28e^{\frac{-3\beta}{\alpha}} + 2e^{\frac{3\beta}{\alpha}} + 4b^2e^{\frac{-\beta}{\alpha}} - 8b^2e^{\frac{-5\beta}{\alpha}}),
\end{aligned}$$

$$\begin{aligned}
a_{41} &= (28e^{\frac{-\beta}{\alpha}} - 4e^{\frac{\beta}{\alpha}} - 18e^{\frac{-3\beta}{\alpha}} - 2b^2e^{\frac{-3\beta}{\alpha}} - 20e^{\frac{-3\beta}{\alpha}} + 8b^2e^{\frac{-5\beta}{\alpha}}), \\
a_{42} &= (6e^{\frac{-3\beta}{\alpha}} - 4e^{\frac{-\beta}{\alpha}} + 12e^{\frac{-5\beta}{\alpha}} - 2b^2e^{\frac{-5\beta}{\alpha}}), \\
a_{43} &= (2e^{\frac{-3\beta}{\alpha}} - 10e^{\frac{-5\beta}{\alpha}}), \\
a_{44} &= (2e^{\frac{-5\beta}{\alpha}}).
\end{aligned}$$

Assuming F^n is a weakly Berwald space, B_m^m equals $hp(1)$. Since α is irrational in (y^i) , the equation (3.2.9) splits into two equations as follows:

$$\beta^2 F_1 B_m^m + \beta^3 G_1 r_{00} + \alpha^2 H_1 s_0 + \alpha^2 \beta^2 I_1 r_0 = 0. \quad (3.2.10)$$

$$\beta^2 F_2 B_m^m + \beta G_2 r_{00} + \alpha^2 H_2 s_0 + \alpha^2 \beta^2 I_2 r_0 = 0. \quad (3.2.11)$$

Where,

$$\begin{aligned}
F_1 &= a_0 \alpha^{10} + a_2 \alpha^8 \beta^2 + a_4 \alpha^6 \beta^4 + a_6 \alpha^4 \beta^6 + a_8 \alpha^2 \beta^8 + a_{10} \beta^{10}, \\
G_1 &= a_{13} \alpha^8 + a_{15} \alpha^6 \beta^2 + a_{17} \alpha^4 \beta^4 + a_{19} \alpha^2 \beta^6 + a_{21} \beta^8, \\
H_1 &= a_{23} \alpha^{10} + a_{25} \alpha^8 \beta^2 + a_{27} \alpha^6 \beta^4 + a_{29} \alpha^4 \beta^6 + a_{31} \alpha^2 \beta^8 + a_{33} \beta^{10}, \\
I_1 &= a_{35} \alpha^8 + a_{37} \alpha^6 \beta^2 + a_{39} \alpha^4 \beta^4 + a_{41} \alpha^2 \beta^6 + a_{43} \beta^8, \\
F_2 &= a_1 \alpha^{10} + a_3 \alpha^8 \beta^2 + a_5 \alpha^6 \beta^4 + a_7 \alpha^4 \beta^6 + a_9 \alpha^2 \beta^8 + a_{11} \beta^{10}, \\
G_2 &= a_{12} \alpha^{10} + a_{14} \alpha^8 \beta^2 + a_{16} \alpha^6 \beta^4 + a_{18} \alpha^4 \beta^6 + a_{20} \alpha^2 \beta^8 + a_{22} \beta^{10}, \\
H_2 &= a_{24} \alpha^{10} + a_{26} \alpha^8 \beta^2 + a_{28} \alpha^6 \beta^4 + a_{30} \alpha^4 \beta^6 + a_{32} \alpha^2 \beta^8 + a_{34} \beta^{10}, \\
I_2 &= a_{36} \alpha^8 + a_{38} \alpha^6 \beta^2 + a_{40} \alpha^4 \beta^4 + a_{42} \alpha^2 \beta^6 + a_{44} \beta^8,
\end{aligned}$$

Eliminating B_m^m from these equations, we obtain

$$\beta R r_{00} + \alpha^2 S s_0 + \alpha^2 \beta^2 T r_0 = 0. \quad (3.2.12)$$

Where,

$$R = \beta^2 F_2 G_1 - F_1 G_2;$$

$$S = F_2 H_1 - F_1 H_2;$$

$$T = F_2 I_1 - F_1 I_2.$$

Since only the terms $\alpha^2 \{ \alpha^{10} (-2b^2 e^{\frac{3\beta}{\alpha}} - 4b^2 e^{\frac{\beta}{\alpha}} + 2b^2 e^{\frac{5\beta}{\alpha}} + 2b^4 e^{\frac{5\beta}{\alpha}} - 6b^4 e^{\frac{3\beta}{\alpha}} + 8b^4 e^{\frac{-\beta}{\alpha}}) \times$

$\alpha^{10}(-8e^{\frac{5\beta}{\alpha}} + 32b^2e^{\frac{3\beta}{\alpha}} - 12b^2e^{\frac{5\beta}{\alpha}} + 22b^4e^{\frac{3\beta}{\alpha}} - 32b^4e^{\frac{\beta}{\alpha}} - 4b^4e^{\frac{5\beta}{\alpha}} + 2e^{\frac{3\beta}{\alpha}} - 8b^2e^{\frac{\beta}{\alpha}} + 8b^4e^{\frac{-\beta}{\alpha}}) -$
 $\alpha^{10}((n(2e^{\frac{5\beta}{\alpha}} + 4e^{\frac{3\beta}{\alpha}} + 2b^2e^{\frac{5\beta}{\alpha}} - 6b^2e^{\frac{\beta}{\alpha}} + 2e^{\frac{\beta}{\alpha}}) + 2e^{\frac{5\beta}{\alpha}} + 4e^{\frac{3\beta}{\alpha}} + 16b^2e^{\frac{\beta}{\alpha}} + 2e^{\frac{\beta}{\alpha}} + 16b^2e^{\frac{3\beta}{\alpha}} +$
 $20b^4e^{\frac{3\beta}{\alpha}} + 8b^4e^{\frac{-3\beta}{\alpha}} + 12b^4e^{\frac{\beta}{\alpha}} - 4b^2e^{\frac{-\beta}{\alpha}} - 2b^2e^{\frac{5\beta}{\alpha}} - 2b^4e^{\frac{5\beta}{\alpha}} - 8b^4e^{\frac{-\beta}{\alpha}}) \times \alpha^{10}(2e^{\frac{5\beta}{\alpha}} + 4b^2e^{\frac{5\beta}{\alpha}} -$
 $8b^2e^{\frac{3\beta}{\alpha}} + 2b^4e^{\frac{5\beta}{\alpha}} - 8b^4e^{\frac{3\beta}{\alpha}} + 8b^4e^{\frac{\beta}{\alpha}})_{s_0}$ of Ss_0 in (3.2.12) do not contain β , We need to have $hp(22)V_{22}$ so that

$$\alpha^{22}s_0 = \beta V_{22}.$$

Case 1:

First we are concerned with $\alpha^2 \not\equiv 0 \pmod{\beta}$ and $b^2 \neq 0$ shows the existence of a function $k(x)$ satisfying $V_{22} = k\alpha^{22}$, and hence $s_0 = k\beta$ then (3.2.12) is reduced to

$$Rr_{00} + \alpha^2 kS + \alpha^2 \beta Tr_0 = 0.$$

So the terms $\{-\alpha^{10}(2e^{\frac{5\beta}{\alpha}} + 4b^2e^{\frac{5\beta}{\alpha}} - 8b^2e^{\frac{3\beta}{\alpha}} + 2b^4e^{\frac{5\beta}{\alpha}} - 8b^4e^{\frac{3\beta}{\alpha}} + 8b^4e^{\frac{\beta}{\alpha}}) \times \alpha^{10}(n(-e^{\frac{5\beta}{\alpha}} -$
 $e^{\frac{3\beta}{\alpha}} - b^2e^{\frac{5\beta}{\alpha}} + b^2e^{\frac{3\beta}{\alpha}} + 2b^2) - e^{\frac{5\beta}{\alpha}} - e^{\frac{3\beta}{\alpha}} - 2b^2e^{\frac{5\beta}{\alpha}} - 6b^2e^{\frac{3\beta}{\alpha}} + 2b^2 + 12b^4e^{\frac{3\beta}{\alpha}})\}r_{00} + \{(\alpha^{10}$
 $(-2b^2e^{\frac{3\beta}{\alpha}} - 4b^2e^{\frac{\beta}{\alpha}} + 2b^2e^{\frac{5\beta}{\alpha}} + 2b^4e^{\frac{5\beta}{\alpha}} - 6b^4e^{\frac{3\beta}{\alpha}} + 8b^4e^{\frac{-\beta}{\alpha}}) \times \alpha^{10}(-8e^{\frac{5\beta}{\alpha}} + 32b^2e^{\frac{3\beta}{\alpha}} - 12b^2$
 $e^{\frac{5\beta}{\alpha}} + 22b^4e^{\frac{3\beta}{\alpha}} - 32b^4e^{\frac{\beta}{\alpha}} - 4b^4e^{\frac{5\beta}{\alpha}} + 2e^{\frac{3\beta}{\alpha}} - 8b^2e^{\frac{\beta}{\alpha}} + 8b^4e^{\frac{-\beta}{\alpha}}) - \alpha^{10}(n(2e^{\frac{5\beta}{\alpha}} + 4e^{\frac{3\beta}{\alpha}} + 2b^2e^{\frac{5\beta}{\alpha}} -$
 $6b^2e^{\frac{\beta}{\alpha}} + 2e^{\frac{\beta}{\alpha}}) + 2e^{\frac{5\beta}{\alpha}} + 4e^{\frac{3\beta}{\alpha}} + 16b^2e^{\frac{\beta}{\alpha}} + 2e^{\frac{\beta}{\alpha}} + 16b^2e^{\frac{3\beta}{\alpha}} + 20b^4e^{\frac{3\beta}{\alpha}} + 8b^4e^{\frac{-3\beta}{\alpha}} + 12b^4e^{\frac{\beta}{\alpha}} -$
 $4b^2e^{\frac{-\beta}{\alpha}} - 2b^2e^{\frac{5\beta}{\alpha}} - 2b^4e^{\frac{5\beta}{\alpha}} - 8b^4e^{\frac{-\beta}{\alpha}}) \times \alpha^{10}(2e^{\frac{5\beta}{\alpha}} + 4b^2e^{\frac{5\beta}{\alpha}} - 8b^2e^{\frac{3\beta}{\alpha}} + 2b^4e^{\frac{5\beta}{\alpha}} - 8b^4e^{\frac{3\beta}{\alpha}} +$
 $8b^4e^{\frac{\beta}{\alpha}})\}\alpha^2 k$

of the above do not contain β . So there must exist $hp(1)U_1$ satisfying $-(2e^{\frac{5\beta}{\alpha}} + 4b^2e^{\frac{5\beta}{\alpha}} -$
 $8b^2e^{\frac{3\beta}{\alpha}} + 2b^4e^{\frac{5\beta}{\alpha}} - 8b^4e^{\frac{3\beta}{\alpha}} + 8b^4e^{\frac{\beta}{\alpha}}) \times (n(-e^{\frac{5\beta}{\alpha}} - e^{\frac{3\beta}{\alpha}} - b^2e^{\frac{5\beta}{\alpha}} + b^2e^{\frac{3\beta}{\alpha}} + 2b^2) - e^{\frac{5\beta}{\alpha}} - e^{\frac{3\beta}{\alpha}} -$
 $2b^2e^{\frac{5\beta}{\alpha}} - 6b^2e^{\frac{3\beta}{\alpha}} + 2b^2 + 12b^4e^{\frac{3\beta}{\alpha}})\}r_{00} + \{(-2b^2e^{\frac{3\beta}{\alpha}} - 4b^2e^{\frac{\beta}{\alpha}} + 2b^2e^{\frac{5\beta}{\alpha}} + 2b^4e^{\frac{5\beta}{\alpha}} - 6b^4e^{\frac{3\beta}{\alpha}} +$
 $8b^4e^{\frac{-\beta}{\alpha}}) \times (-8e^{\frac{5\beta}{\alpha}} + 32b^2e^{\frac{3\beta}{\alpha}} - 12b^2e^{\frac{5\beta}{\alpha}} + 22b^4e^{\frac{3\beta}{\alpha}} - 32b^4e^{\frac{\beta}{\alpha}} - 4b^4e^{\frac{5\beta}{\alpha}} + 2e^{\frac{3\beta}{\alpha}} - 8b^2e^{\frac{\beta}{\alpha}} +$
 $8b^4e^{\frac{-\beta}{\alpha}}) - (n(2e^{\frac{5\beta}{\alpha}} + 4e^{\frac{3\beta}{\alpha}} + 2b^2e^{\frac{5\beta}{\alpha}} - 6b^2e^{\frac{\beta}{\alpha}} + 2e^{\frac{\beta}{\alpha}}) + 2e^{\frac{5\beta}{\alpha}} + 4e^{\frac{3\beta}{\alpha}} + 16b^2e^{\frac{\beta}{\alpha}} + 2e^{\frac{\beta}{\alpha}} + 16b^2e^{\frac{3\beta}{\alpha}}$
 $+ 20b^4e^{\frac{3\beta}{\alpha}} + 8b^4e^{\frac{-3\beta}{\alpha}} + 12b^4e^{\frac{\beta}{\alpha}} - 4b^2e^{\frac{-\beta}{\alpha}} - 2b^2e^{\frac{5\beta}{\alpha}} - 2b^4e^{\frac{5\beta}{\alpha}} - 8b^4e^{\frac{-\beta}{\alpha}})k\}\alpha^2 = \beta U_1$. It is
 a contradiction, which leads to $k=0$. Hence we obtain $s_0 = 0; s_j = 0$. Substituting $s_0 = 0$
 into (3.2.12), we have

$$Rr_{00} + \alpha^2 \beta T r_0 = 0. \quad (3.2.13)$$

Then only terms $\beta^3 \{(\beta^{10}(2e^{\frac{-5\beta}{\alpha}}) \times (\beta^8[(n(5e^{\frac{-5\beta}{\alpha}} + 6e^{\frac{-3\beta}{\alpha}}) + 4e^{\frac{-5\beta}{\alpha}} + 12e^{\frac{-3\beta}{\alpha}} - 2e^{\frac{-\beta}{\alpha}} - b^2e^{\frac{-5\beta}{\alpha}})]\}r_{00}$ of (3.2.13) does not contain α^2 , and hence we must have $hp(21)V_{21}$ such that $\beta^{21}r_{00} = \alpha^2V_{21}$. From $\alpha^2 \not\equiv 0 \pmod{\beta}$ there exists a function $f(x)$ such that

$$r_{00} = \alpha^2 f(x); r_{ij} = a_{ij} f(x). \quad (3.2.14)$$

Transvecting (3.2.14) by $b^i y^j$, we have

$$r_0 = \beta f(x); r_j = b_j f(x). \quad (3.2.15)$$

Substituting (3.2.14) and (3.2.15) into (3.2.13), we have

$$f(x)(R + \beta^2 T) = 0. \quad (3.2.16)$$

Let us assume $f(x) \neq 0$. Then (3.2.16) implies

$$\left\{ \begin{array}{l} \{-(2e^{\frac{5\beta}{\alpha}} + 4b^2e^{\frac{5\beta}{\alpha}} - 8b^2e^{\frac{3\beta}{\alpha}} + 2b^4e^{\frac{5\beta}{\alpha}} - 8b^4e^{\frac{3\beta}{\alpha}} + 8b^4e^{\frac{\beta}{\alpha}}) \times (n(-e^{\frac{5\beta}{\alpha}} - e^{\frac{3\beta}{\alpha}} \\ -b^2e^{\frac{5\beta}{\alpha}} + b^2e^{\frac{3\beta}{\alpha}} + 2b^2) - e^{\frac{5\beta}{\alpha}} - e^{\frac{3\beta}{\alpha}} - 2b^2e^{\frac{5\beta}{\alpha}} - 6b^2e^{\frac{3\beta}{\alpha}} + 2b^2 + 12b^4e^{\frac{3\beta}{\alpha}})\} \alpha^{20} = \beta V_{19}, \end{array} \right.$$

Where V_{19} is $hp(19)$. This implies $V_{19} = 0$, provided that $b^2 \neq 0$. Hence $f(x) = 0$ must hold and we obtain

$$r_{00} = 0; r_{ij} = 0 \text{ and } r_0 = 0; r_j = 0.$$

Conversely, substituting $r_{00} = 0, s_0 = 0$ and $r_0 = 0$ into (3.2.9), we have $B_m^m = 0$. That is, the Finsler space with (3.1.1) is a weakly-Berwald space.

On the other hand, we suppose that the Finsler space with (3.1.1) is a Berwald space. Then we have $r_{00} = 0, s_0 = 0$ and $r_0 = 0$, because the space is a weakly-Berwald space from the

above discussion. Substituting the above into (3.2.1), we have $B^m = 0$, that is, the Finsler space with (3.1.1) is a Berwald space. Hence $s_{ij} = 0$ holds.

Case2:

Consider $\alpha^2 \equiv 0 \pmod{\beta}$. Lemma (3.1.2) indicates that, for $n = 2$, $b^2 = 0$ and $\alpha^2 = \beta\delta, \delta = d_i(x)y^i$. The following conditions (3.2.12) are rewritten in the form

$$R'_{r00} + \delta S'_0 = 0, \quad (3.2.17)$$

where,

$$\begin{aligned} R' = & \beta^2 [\{\beta^5 \delta^5 (-8e^{\frac{5\beta}{\alpha}} + 2e^{\frac{3\beta}{\alpha}}) + \beta^6 \delta^4 (-44e^{\frac{3\beta}{\alpha}} + 4e^{\frac{5\beta}{\alpha}} + 16e^{\frac{\beta}{\alpha}}) + \beta^7 \delta^3 (-68e^{\frac{\beta}{\alpha}} + 8e^{\frac{3\beta}{\alpha}} + 44e^{\frac{-\beta}{\alpha}}) + \beta^8 \delta^2 (-18e^{\frac{-\beta}{\alpha}} - 8e^{\frac{\beta}{\alpha}} + 48e^{\frac{-3\beta}{\alpha}} + 2e^{\frac{3\beta}{\alpha}}) + \beta^9 \delta (8e^{\frac{-3\beta}{\alpha}} - 4e^{\frac{-\beta}{\alpha}}) + \beta^{10} (2e^{\frac{-5\beta}{\alpha}})\} \times \\ & \{\beta^4 \delta^4 (n(5e^{\frac{5\beta}{\alpha}} + 3e^{\frac{3\beta}{\alpha}}) + 5e^{\frac{3\beta}{\alpha}} + 3e^{\frac{\beta}{\alpha}}) + \beta^5 \delta^3 (n(20e^{\frac{3\beta}{\alpha}} + 22e^{\frac{\beta}{\alpha}} + 3e^{\frac{5\beta}{\alpha}}) - e^{\frac{3\beta}{\alpha}} + 35e^{\frac{\beta}{\alpha}} + 3e^{\frac{5\beta}{\alpha}}) + \beta^6 \delta^2 (n(52e^{\frac{-\beta}{\alpha}} + 23e^{\frac{\beta}{\alpha}} + 11e^{\frac{3\beta}{\alpha}} - 2e^{\frac{5\beta}{\alpha}} - 2e^{\frac{3\beta}{\alpha}}) + 49e^{\frac{-\beta}{\alpha}} + 56e^{\frac{\beta}{\alpha}} - 24e^{\frac{3\beta}{\alpha}}) + \beta^7 \delta (n(42e^{\frac{-3\beta}{\alpha}} + 14e^{\frac{-\beta}{\alpha}}) + 15e^{\frac{-3\beta}{\alpha}} + 48e^{\frac{-\beta}{\alpha}} - 10e^{\frac{\beta}{\alpha}}) + \beta^8 (n(5e^{\frac{-5\beta}{\alpha}} + 6e^{\frac{-3\beta}{\alpha}}) + 4e^{\frac{-5\beta}{\alpha}} + 12e^{\frac{-3\beta}{\alpha}} - 2e^{\frac{-\beta}{\alpha}}) \} - \{(\beta^5 \delta^5 (2e^{\frac{5\beta}{\alpha}}) + \beta^6 \delta^4 (8e^{\frac{5\beta}{\alpha}} + 8e^{\frac{3\beta}{\alpha}}) + \beta^7 \delta^3 (44e^{\frac{3\beta}{\alpha}} - 8e^{\frac{\beta}{\alpha}} - 8e^{\frac{5\beta}{\alpha}}) + \beta^8 \delta^2 (74e^{\frac{\beta}{\alpha}} - 60e^{\frac{-\beta}{\alpha}} - 20e^{\frac{3\beta}{\alpha}} + 2e^{\frac{5\beta}{\alpha}}) + \beta^9 \delta (40e^{\frac{-\beta}{\alpha}} - 58e^{\frac{-3\beta}{\alpha}} - 4e^{\frac{\beta}{\alpha}}) + \beta^{10} (2e^{\frac{-3\beta}{\alpha}} - 12e^{\frac{-5\beta}{\alpha}}) \times [\beta^5 \delta^5 (n(-e^{\frac{5\beta}{\alpha}} - e^{\frac{3\beta}{\alpha}}) - e^{\frac{5\beta}{\alpha}} - e^{\frac{3\beta}{\alpha}})) + \beta^6 \delta^4 (n(-8e^{\frac{\beta}{\alpha}} - 7e^{\frac{3\beta}{\alpha}} - 8e^{\frac{5\beta}{\alpha}}) - 6e^{\frac{\beta}{\alpha}} - 7e^{\frac{5\beta}{\alpha}}) + \beta^7 \delta^3 (n(-26e^{\frac{3\beta}{\alpha}} - 24e^{\frac{\beta}{\alpha}} - 22e^{\frac{-\beta}{\alpha}} + 3e^{\frac{5\beta}{\alpha}}) + 19e^{\frac{3\beta}{\alpha}} - 67e^{\frac{\beta}{\alpha}} - 8e^{\frac{-\beta}{\alpha}}) + \beta^8 \delta^2 ((n(-41e^{\frac{-\beta}{\alpha}} - 14e^{\frac{\beta}{\alpha}} - 24e^{\frac{-3\beta}{\alpha}} + 2e^{\frac{3\beta}{\alpha}}) - 80e^{\frac{-\beta}{\alpha}} - 8e^{\frac{\beta}{\alpha}} + 8e^{\frac{3\beta}{\alpha}}) + \beta^9 \delta ((n(-21e^{\frac{-3\beta}{\alpha}} - 9e^{\frac{-5\beta}{\alpha}}) - 24e^{\frac{-3\beta}{\alpha}} + 3e^{\frac{-5\beta}{\alpha}} - 8e^{\frac{-\beta}{\alpha}} + 2e^{\frac{\beta}{\alpha}}) + \beta^{10} (2ne^{\frac{-5\beta}{\alpha}}))\} \\ S' = & [\{\beta^6 \delta^4 ((n(-8e^{\frac{5\beta}{\alpha}} - 14e^{\frac{3\beta}{\alpha}} - 10e^{\frac{\beta}{\alpha}}) - 12e^{\frac{5\beta}{\alpha}} - 8e^{\frac{3\beta}{\alpha}} - 10e^{\frac{\beta}{\alpha}}) + \beta^7 \delta^3 ((n(-34e^{\frac{3\beta}{\alpha}} - 58e^{\frac{\beta}{\alpha}} + 2e^{\frac{5\beta}{\alpha}} - 40e^{\frac{-\beta}{\alpha}}) + 2e^{\frac{3\beta}{\alpha}} - 18e^{\frac{\beta}{\alpha}} + 4e^{\frac{5\beta}{\alpha}} - 52e^{\frac{-\beta}{\alpha}}) + \beta^8 \delta^2 ((n(-70e^{\frac{-\beta}{\alpha}} - 28e^{\frac{\beta}{\alpha}} + 4e^{\frac{3\beta}{\alpha}} - 78e^{\frac{-3\beta}{\alpha}}) - 10e^{\frac{-\beta}{\alpha}} + 144e^{\frac{\beta}{\alpha}} - 74e^{\frac{3\beta}{\alpha}} - 142e^{\frac{-3\beta}{\alpha}} + 124e^{\frac{-\beta}{\alpha}} - 2e^{\frac{5\beta}{\alpha}}) + \beta^9 \delta ((n(-38e^{\frac{-3\beta}{\alpha}} - 36e^{\frac{-5\beta}{\alpha}}) - 250e^{\frac{-3\beta}{\alpha}} - 40e^{\frac{-5\beta}{\alpha}} + 84e^{\frac{-\beta}{\alpha}} + 30e^{\frac{\beta}{\alpha}} - 8e^{\frac{3\beta}{\alpha}}) + \beta^{10} (n(-4e^{\frac{-5\beta}{\alpha}}) + 8e^{\frac{-5\beta}{\alpha}} - 58e^{\frac{-3\beta}{\alpha}} + 8e^{\frac{-\beta}{\alpha}})) \} \times \\ & (\beta^5 \delta^5 (-8e^{\frac{5\beta}{\alpha}} + 2e^{\frac{3\beta}{\alpha}}) + \beta^6 \delta^4 (-44e^{\frac{3\beta}{\alpha}} + 4e^{\frac{5\beta}{\alpha}} + 16e^{\frac{\beta}{\alpha}}) + \beta^7 \delta^3 (-68e^{\frac{\beta}{\alpha}} + 8e^{\frac{3\beta}{\alpha}} + 44e^{\frac{-\beta}{\alpha}}) + \beta^8 \delta^2 (-18e^{\frac{-\beta}{\alpha}} - 8e^{\frac{\beta}{\alpha}} + 48e^{\frac{-3\beta}{\alpha}} + 2e^{\frac{3\beta}{\alpha}}) + \beta^9 \delta (8e^{\frac{-3\beta}{\alpha}} - 4e^{\frac{-\beta}{\alpha}}) + \beta^{10} (2e^{\frac{-5\beta}{\alpha}})) - [\{\beta^5 \delta^5 (n(2e^{\frac{5\beta}{\alpha}} + 4e^{\frac{3\beta}{\alpha}} + 2e^{\frac{\beta}{\alpha}}) + 2e^{\frac{5\beta}{\alpha}} + 4e^{\frac{3\beta}{\alpha}} + 2e^{\frac{\beta}{\alpha}}) + \beta^6 \delta^4 (n(26e^{\frac{\beta}{\alpha}} + 8e^{\frac{5\beta}{\alpha}} + 24e^{\frac{3\beta}{\alpha}} + 14e^{\frac{-\beta}{\alpha}}) + 4e^{\frac{\beta}{\alpha}} + 18e^{\frac{5\beta}{\alpha}} + 20e^{\frac{3\beta}{\alpha}}) + \beta^7 \delta^3 (n(14e^{\frac{3\beta}{\alpha}} + 58e^{\frac{\beta}{\alpha}} - 4e^{\frac{5\beta}{\alpha}} + 70e^{\frac{-\beta}{\alpha}} + 30e^{\frac{-3\beta}{\alpha}}) + 38e^{\frac{3\beta}{\alpha}} - 58e^{\frac{\beta}{\alpha}} - 10e^{\frac{5\beta}{\alpha}} + 49e^{\frac{-\beta}{\alpha}} + 38e^{\frac{-3\beta}{\alpha}}) + \beta^8 \delta^2 (n(28e^{\frac{-\beta}{\alpha}} + 86e^{\frac{-3\beta}{\alpha}} + 18e^{\frac{-5\beta}{\alpha}}) - 30e^{\frac{-\beta}{\alpha}} + 267e^{\frac{-3\beta}{\alpha}} + 6e^{\frac{-5\beta}{\alpha}} - 126e^{\frac{\beta}{\alpha}} + 38e^{\frac{3\beta}{\alpha}}) + \beta^9 \delta (n(4e^{\frac{-3\beta}{\alpha}} + 22e^{\frac{-5\beta}{\alpha}}) + 168e^{\frac{-3\beta}{\alpha}} + 12e^{\frac{-5\beta}{\alpha}} - 56e^{\frac{-\beta}{\alpha}} - 8e^{\frac{\beta}{\alpha}}) + \beta^{10} (-2e^{\frac{-5\beta}{\alpha}} + 8e^{\frac{-3\beta}{\alpha}})) \} \times \end{aligned}$$

$$\{(\beta^5 \delta^5 (2e^{\frac{5\beta}{\alpha}}) + \beta^6 \delta^4 (8e^{\frac{5\beta}{\alpha}} + 8e^{\frac{3\beta}{\alpha}}) + \beta^7 \delta^3 (44e^{\frac{3\beta}{\alpha}} - 8e^{\frac{\beta}{\alpha}} - 8e^{\frac{5\beta}{\alpha}}) + \beta^8 \delta^2 (74e^{\frac{\beta}{\alpha}} - 60e^{\frac{-\beta}{\alpha}} - 20e^{\frac{3\beta}{\alpha}} + 2e^{\frac{5\beta}{\alpha}}) + \beta^9 \delta (40e^{\frac{-\beta}{\alpha}} - 58e^{\frac{-3\beta}{\alpha}} - 4e^{\frac{\beta}{\alpha}}) + \beta^{10} (2e^{\frac{-3\beta}{\alpha}} - 12e^{\frac{-5\beta}{\alpha}}))\}.$$

Since the only term which does not contain δ is $\beta^2\{(\beta^{10}(2e^{\frac{-5\beta}{\alpha}}) \times \beta^8(n(5e^{\frac{-5\beta}{\alpha}} + 6e^{\frac{-3\beta}{\alpha}}) + 4e^{\frac{-5\beta}{\alpha}} + 12e^{\frac{-3\beta}{\alpha}} - 2e^{\frac{-\beta}{\alpha}}) - \beta^{10}(2e^{\frac{-3\beta}{\alpha}} - 12e^{\frac{-5\beta}{\alpha}}) \times \beta^{10}(2ne^{\frac{-5\beta}{\alpha}})\}r_{00}$ in (3.2.17), We require $hp(1)v_1$ so that $r_{00} = \delta v_1$. We have $s_0 = 0; s_j = 0$, now (3.2.17) becomes

$$R' r_{00} = 0, \quad (3.2.18)$$

which indicates that

$$r_{00} = 0; r_{ij} = 0 \text{ and } r_0 = 0; r_j = 0.$$

Using $r_{00} = 0$, $r_0 = 0$, and $s_0 = 0$, we may conclude that $B_m^m = 0$. The space containing (3.1.1) is a weakly-Berwald space. Thus, we state the following.

Theorem 3.2.1. *A Finsler space endowed with the metric (3.1.1) is weakly Berwald space if and only if the following are satisfied:*

- (1) $\alpha^2 \not\equiv 0 \pmod{\beta}$, which yields $r_{ij} = 0$ and $s_j = 0$.
- (2) $\alpha^2 \equiv 0 \pmod{\beta}$ indicates that $n=2$, $b^2 = 0$, and $r_{ij} = 0, s_j = 0$ are satisfied, where $\alpha^2 = \beta \delta, \delta = d_i y^i$.

CHAPTER 4

Projectively Flat of Finsler-Metric with Constant Flag Curvature

This chapter examines general (α, β) -metrics, a significant class of Finsler metrics. The flag curvature of generic (α, β) -metrics, as well as their geometric characteristics and theoretical implications, are examined in this chapter. Classifying locally projectively flat cubic (α, β) -metric and generalized Matsumoto metrics with constant Ricci curvature is the goal of this chapter. The objective of this research is to construct a generalized Matsumoto and a cubic (α, β) -metric with zero flag curvature.

4.1 Introduction

One of the most essential challenges in Finsler geometry is to examine and characterize Finsler metrics with constant flag curvature. Another task is to investigate and characterize projectively flat Finsler metrics on an open domain in R^n . The latter is the famous Hilbert's Fourth Problem in the regular case. In Riemannian geometry, these two problems are the essentially same. The Beltrami theorem tells us that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. However, there are locally projectively flat Finsler metrics that are not of constant flag curvature; and there are Finsler metrics of constant flag curvature that are not locally projectively flat.

Distance functions induced by Finsler metrics are regarded as smooth ones. Thus Hilbert's Fourth Problem in the smooth case is to characterize Finsler metrics on an open subset in R^n whose geodesics are straight lines. Such Finsler metrics are called projective Finsler metrics. In Finsler geometry, the flag curvature $K(P, y)$ is an analogue of the sectional curvature in Riemannian geometry. It is known that every projective Finsler metric is of scalar curvature, namely, the flag curvature $K(P, y) = K(y)$ is a scalar function of tangent vectors

y. In the early 20th century, L. Berwald studied projective Finsler metrics of constant flag curvature, particularly in the case of zero flag curvature [20, 22]. P. Funk classified all projective Finsler metrics with constant flag curvature on convex domains in R^2 [50, 51]. Later on, Funk tried to show the uniqueness of projectively flat Finsler metrics with $K = 1$ on S^2 , with an additional condition, he showed that the standard Riemannian metric is the only such metric. In 1995-1996, R. Bryant showed that there is exactly a 2-parameter family of projectively flat Finsler metrics on S^2 with $K = 1$ [24, 25].

There is another important quantity closely related to the flag curvature, the so-called S -curvature. The S -curvature, a scalar function on TM , was introduced by the second author to study volume comparison in Riemann–Finsler geometry [126, 128]. The Finsler metric F is said to be of isotropic S -curvature if $S = (n + 1)cF$, where c is a scalar function on M . Further, if c is a constant, then F is said to be of constant S -curvature. It is proved that for a Finsler metric F of scalar flag curvature, if it is of isotropic S -curvature, then F must be of almost isotropic flag curvature with flag curvature (see [30]). Thus, the flag curvature and the S -curvature are closely related.

4.1.1 Preliminaries

Let F be a Finsler metric on a manifold M . In a standard local coordinate system (x^i, y^i) in TM , $F = F(x, y)$ is a function of (x^i, y^i) . Let

$$g_{ij}(x, y) = \frac{1}{2}[F^2]_{y^i y^j}(x, y),$$

and $(g^{ij}) = (g_{ij})^{-1}$. For a nonzero vector $y = y^i \frac{\partial}{\partial x^i}|_p \in T_p M$, we obtain an inner product on $T_p M$,

$$g_y(u, v) = g_{ij}(x, y)u^i v^j, u = u^i \frac{\partial}{\partial x^i}|_p, v = v^i \frac{\partial}{\partial x^i}|_p,$$

which satisfies that $g_y(y, y) = F^2(y)$. Let

$$h_y(u, v) = g_y(u, v) - F^{-2}(y)g_y(u, y)g_y(v, y), \quad u, v \in T_p M.$$

h_y is called an angular form on T_pM associated with y .

The geodesics of F are characterized locally by [125];

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0,$$

The second variation of geodesics yields a family of linear maps $R_y = R_k^i(x, y)dx^k \otimes \frac{\partial}{\partial x^i}|_p : T_pM \rightarrow T_pM$, where $R_k^i = R_k^i(x, y)$ are given by

$$R_k^i = 2 \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}. \quad (4.1.1)$$

For any non-zero $y \in T_pM$, R_y satisfies $R_y(y) = 0$, $g_y(R_y(u), v) = g_y(u, R_y(v))$, $u, v \in T_pM$.

The Ricci curvature $\text{Ric}(y)$ is defined to be the trace of R_y ,

$$\text{Ric}(y) = R_m^m(x, y).$$

One can easily verify that for any tangent plane $P = \text{span}\{y, u\} \subset T_pM$, the following quantity is well-defined:

$$K(P, y) = \frac{g_y(R_y(u), u)}{F^2(y)h_y(u, u)},$$

$K(P, y)$ is called the flag curvature of the flag $\{P, y\}$.

A Finsler metric F is said to be of constant flag curvature $K = \lambda$, if for any $y \in P \subset T_pM$,

$$K(P, y) = \lambda.$$

This corresponds to the following identity [125]:

$$R_k^i = \lambda \{F^2 \delta_k^i - F F_{y^k} y^i\}. \quad (4.1.2)$$

If (4.1.2) holds, then $R_m^m = (n-1)\lambda F^2$. That is, if $K = \lambda$, then $\text{Ric} = (n-1)\lambda F^2$.

4.1.2 Equations of Geodesics of Projectively Flat Cubic (α, β) -Metric

In this section, we will find the equations of geodesics for projectively flat cubic (α, β) -metric with constant Ricci curvature. Let $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ be a cubic (α, β) -metric on a manifold M .

For a Finsler space with (α, β) -metric, the geodesic spray coefficients $G^i(x, y)$ are of the form

$$G^i = \bar{G}^i + D^i,$$

where D^i is represented as

$$D^i = \left(\frac{\eta}{\alpha^2} \right) \left\{ y^i + \left(\frac{\alpha^3 F F_{11}}{\beta^2 F_1 F_2} \right) p^i \right\} + \left(\frac{\alpha F_2}{F_1} \right) s_0^i, \quad (4.1.3)$$

and

$$\eta = \left(\frac{\alpha^2 F_2}{2F} \right) (r_{00} - 2\xi),$$

$$\xi = \left(\frac{F^3}{\alpha^2 T} \right) \left\{ \left(\frac{\gamma^2 F_{11}}{2\beta^2} \right) r_{00} + F_2 s_0 \right\},$$

$$T = \left(\frac{F}{\alpha} \right)^3 \left(F_1 + \frac{\gamma^2 F_{22}}{\alpha} \right).$$

Now we shall find D^i for a projectively flat cubic (α, β) -metric. For the cubic (α, β) -metric $F = \frac{(\alpha + \beta)^3}{\alpha^2}$,

$$\begin{aligned} F_1 &= \frac{(\alpha + \beta)^2(\alpha - 2\beta)}{\alpha^3}, & F_{11} &= \frac{6(\alpha + \beta)\beta^2}{\alpha^4}, \\ F_2 &= \frac{3(\alpha + \beta)^2}{\alpha^2}, & F_{22} &= \frac{6(\alpha + \beta)}{\alpha^2}. \end{aligned}$$

Where $F_1 = \partial F / \partial \alpha$, $F_{11} = \partial^2 F / \partial \alpha^2$, $F_2 = \partial F / \partial \beta$, $F_{22} = \partial^2 F / \partial \beta^2$. Then

$$T = \frac{(\alpha + \beta)^{10}((\alpha + \beta)(\alpha - 2\beta) + 6\gamma^2)}{\alpha^{12}},$$

$$\xi = \frac{(3\gamma^2 r_{00} + 3\alpha^2(\alpha + \beta)s_0)}{((\alpha + \beta)(\alpha - 2\beta) + 6\gamma^2)},$$

$$\eta = \frac{3\alpha^2(r_{00}(\alpha - 2\beta) - 6\alpha^2 s_0)}{2((\alpha + \beta)(\alpha - 2\beta) + 6(\Delta - \beta^2))}.$$

Substituting these in equation (4.1.3), we obtain D^i as follows:

$$D^i = \frac{3\{r_{00}(\alpha - 2\beta) - 6\alpha^2 s_0\} \times \{y^i + \frac{2\alpha^2}{\alpha - 2\beta} p^i\}}{2((\alpha + \beta)(\alpha - 2\beta) + 6(\Delta - \beta^2))} + (\frac{3\alpha^2}{\alpha - 2\beta}) s_0^i. \quad (4.1.4)$$

Where $p^i = B^i - \frac{\beta}{\alpha^2} y^i$, $\Delta = \gamma^2 + \beta^2$.

For projectively flat cubic (α, β) -metric, we have

$$s_0^i = \frac{36s_0 b^2 B^i + \beta s_i}{6b^2(1 + 6b^2)}.$$

Then D^i can be written as

$$D^i = \frac{\{(r_{00}P_2 + s_0P_3)y^i + (r_{00}P_4 + s_0P_5)B^i + s^iP_6\}}{P_1}. \quad (4.1.5)$$

Where,

$$P_1 = 12b^2(1 + 6b^2)(\alpha^3 - 3\alpha^2\beta - 6\alpha\beta^2 + 6b^2\alpha^3 + 16\beta^3 - 12b^2\alpha^2\beta),$$

$$P_2 = 6b^2(1 + 6b^2)(3\alpha^2 - 18\alpha\beta + 24\beta^2),$$

$$P_3 = -18\alpha^3 + 72\alpha^2\beta,$$

$$P_4 = 36\alpha^3b^2 - 72\alpha^2\beta b^2 + 216\alpha^3b^4 - 432\alpha^2\beta b^4,$$

$$P_5 = -216\alpha^3\beta b^2 - 1728\alpha^2\beta^2b^2,$$

$$P_6 = 6\alpha^4\beta - 6\alpha^3\beta^2 - 48\alpha^2\beta^3 + 36\alpha^4\beta b^2,$$

$$B^i = a^{ir}b_r, \quad Y_i = a_{ir}y^r, \quad b^2 = a^{ij}b_ib_j, \quad s^i = a^{ir}s_rs^jy_j, \quad s_i = b_rs_i^r.$$

Lemma 4.1.1. For a projectively flat cubic (α, β) -metric $F = \frac{(\alpha + \beta)^3}{\alpha^2}$, D^i is given by (4.1.5).

If we define [10]

$$K_{jk}^i = \frac{a_{jk}s^i + r_{jk}B^i}{b^2},$$

then D^i is given by

$$D^i = \frac{\{(r_{00}P_2 + s_0P_3)y^i + (r_{00}P'_4 + k_{000}P'_5)B^i + K_{00}^iP'_6\}}{P_1}, \quad (4.1.6)$$

where,

$$P_1 = 12b^2(1 + 6b^2)(\alpha^3 - 3\alpha^2\beta - 6\alpha\beta^2 + 6b^2\alpha^3 + 16\beta^3 - 12b^2\alpha^2\beta),$$

$$P_2 = 6b^2(1 + 6b^2)(3\alpha^2 - 18\alpha\beta + 24\beta^2),$$

$$P_3 = -18\alpha^3 + 72\alpha^2\beta,$$

$$P'_4 = 36\alpha^3b^2 - 108\alpha^2\beta b^2 + 216\alpha^3b^4 - 432\alpha^2\beta b^2 + 216\alpha\beta^2b^2 + 1728\beta^3b^2 - 6\alpha^2\beta + 6\alpha\beta^2 + 48\beta^3,$$

$$P'_5 = -216\alpha\beta b^4 - 1728\beta^2b^4,$$

$$P'_6 = 6\alpha^2\beta b^2 - 6\alpha\beta^2b^2 - 48\beta^3b^2 + 36\alpha^2\beta b^4.$$

Where, $K_{ijk} = a_{jr}K_{ik}^r$.

Substituting (4.1.6) into G^i , we obtain

$$G^i = \bar{G}^i + \frac{\{((r_{00}P_2 + s_0P_3) + (r_{i0}P'_4 + k_{i00}P'_5)B^i + K_{i0}^iP'_6)y^i\}}{P_1}. \quad (4.1.7)$$

Let

$$P = \frac{\{(r_{00}P_2 + s_0P_3) + (r_{i0}P'_4 + k_{i00}P'_5)B^i + K_{i0}^iP'_6\}}{P_1}.$$

Where $P_1, P_2, P_3, P'_4, P'_5, P'_6$ are as defined above. Then (4.1.7) becomes

$$G^i = \bar{G}^i + Py^i,$$

The S-curvature is given by

$$S = (n+1)\{P - dp(y)\}.$$

Substituting P into the above equation, we obtain

$$S = (n+1) \left\{ \frac{\{(r_{00}P_2 + s_0P_3) + (r_{i0}P'_4 + k_{i00}P'_5)B^i + K_{i0}^iP'_6\}}{P_1} - dp(y) \right\}, \quad (4.1.8)$$

where $p := \ln \sqrt{1 - \|\beta\|^2}$.

Theorem 4.1.2. *The S -curvature of a projectively flat cubic (α, β) -metric is given by (4.1.8).*

4.1.3 Projectively Flat Cubic (α, β) -Metric with $K = 0$

Now we study locally projectively flat cubic (α, β) -metric with constant Ricci curvature.

Let

$$\phi = b_{i|j}y^i y^j, \quad \psi = b_{i|j|k}y^i y^j y^k.$$

Since β is closed [58],

$$s_{ij} = \frac{1}{2} (b_{i|j} - b_{j|i}) = 0.$$

Hence $s_i = 0$, and the Riemannian curvature $R_y = R_k^i dx^k \otimes \frac{\partial}{\partial x^i}$ of F is associated with the Riemannian curvature $\bar{R}_y = \bar{R}_k^i dx^k \otimes \frac{\partial}{\partial x^i}$ of α by

$$\bar{R}_k^i = R_k^i + \left\{ \frac{3}{4} \left(\frac{\phi}{F} \right)^2 - \frac{\psi}{2F} \right\} h_k^i + \eta_k y^i, \quad (4.1.9)$$

where,

$$\phi = b_{i|j}y^i y^j,$$

$$\psi = b_{i|j|k}y^i y^j y^k,$$

$$h_k^i = \delta_k^i - l_k l^i,$$

$$\eta_k = -(b_{r|j|k} - b_{r|k|j})y^r y^j.$$

Substituting $k = i$ into (4.1.9), since $h_i^i = n - 1$ and $\eta_i y^i = 0$, we obtain

$$\bar{R}_i^i = R_i^i + \left\{ \frac{3}{4} \left(\frac{\phi}{F} \right)^2 - \frac{\psi}{2F} \right\} (n - 1). \quad (4.1.10)$$

Now from $R_k^r = K(F^2 \delta_k^r - F \frac{\partial F}{\partial y^k} y^r)$ with $r = k = i$ yields

$$\bar{R}_i^i = (n - 1) K F^2, \quad R_i^i = (n - 1) \mu \alpha^2. \quad (4.1.11)$$

Now from (4.1.10),

$$K F^2 = \mu \alpha^2 + \frac{3}{4} \left(\frac{\phi}{F} \right)^2 - \frac{\psi}{2F}. \quad (4.1.12)$$

where $n \neq 1$.

Since β is closed [58], F must be pointwise projectively equivalent to α . Thus α must be locally projectively flat. This implies that α is of constant curvature $\bar{K} = \mu$.

A Finsler metric F is said to be of constant flag curvature $K = \lambda$. Thus from (4.1.12),

$$\lambda F^2 = \mu \alpha^2 + \frac{3}{4} \left(\frac{\phi}{F} \right)^2 - \frac{\psi}{2F}. \quad (4.1.13)$$

Theorem 4.1.3. Let $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ be a locally projectively flat cubic (α, β) -metric, suppose that F has constant Ricci curvature $\text{Ric} = (n - 1)\lambda F^2$, then

(a) F has flag curvature $K = \lambda = 0$,

(b) α has curvature $\bar{K} = \mu = 0$.

Proof. From equation (4.1.13), we obtain

$$\frac{3}{4} \phi^2 = \frac{\lambda (\alpha + \beta)^{12}}{\alpha^8} + \frac{\psi (\alpha + \beta)^3}{\alpha^2} - \frac{\psi (\alpha + \beta)^3}{2\alpha^2} - \frac{\mu (\alpha + \beta)^6}{\alpha^2}. \quad (4.1.14)$$

Further, (4.1.14) is equivalent to

$$\frac{3}{4}\phi^2 = \frac{\lambda(\alpha + \beta)^{12}}{\alpha^8} + \frac{\psi(\alpha + \beta)^3}{\alpha^2}. \quad (4.1.15)$$

$$\frac{\psi}{2} = -\mu(\alpha + \beta)^3. \quad (4.1.16)$$

Substituting (4.1.16) into (4.1.15), we get

$$\frac{3}{4}\phi^2 = \frac{\lambda(\alpha + \beta)^{12}}{\alpha^8} - \frac{2\mu(\alpha + \beta)^6}{\alpha^2}. \quad (4.1.17)$$

Now, differentiating (4.1.17) w.r.t. y^i and y^j , we obtain

$$\frac{3}{2}\phi b_{i|j|k}y^iy^j = \frac{12\lambda(\alpha + \beta)^{11}b_{i|k}y^i}{\alpha^8} - \frac{12\mu(\alpha + \beta)^5b_{i|k}y^i}{\alpha^2}. \quad (4.1.18)$$

Contracting (4.1.18) with y^k , and we have

$$\frac{3}{2}\phi\psi = \frac{12\lambda(\alpha + \beta)^{11}\phi}{\alpha^8} - \frac{12\mu(\alpha + \beta)^5\phi}{\alpha^2}. \quad (4.1.19)$$

Substituting (4.1.16) into (4.1.19), we have

$$-3\mu\alpha^8(\alpha + \beta)^{-8} + 12\mu(\alpha + \beta)^{-6}\alpha^6 = 12\lambda. \quad (4.1.20)$$

Further differentiating (4.1.20) with respect to y^i and y^j , we obtain

$$24\mu\alpha^8(\alpha + \beta)^{-9}b_{i|k}y^i - 72\mu\alpha^6(\alpha + \beta)^{-7}b_{i|k}y^i = 0. \quad (4.1.21)$$

Now again contracting (4.1.21) by y^k and we get;

$$\phi\mu(24\alpha^8 - 72\alpha^6(\alpha + \beta)^2) = 0. \quad (4.1.22)$$

Suppose $\phi(24\alpha^8 - 72\alpha^6(\alpha + \beta)^2) \neq 0$. This implies that

$$\mu = 0.$$

Substituting this into (4.1.17), we obtain

$$\phi^2 = \frac{4\lambda(\alpha + \beta)^{12}}{3\alpha^8} \quad (4.1.23)$$

We observe that $\lambda \geq 0$ and

$$\phi = \varepsilon \frac{2\sqrt{\lambda}(\alpha + \beta)^6}{\sqrt{3}\alpha^4}, \quad (4.1.24)$$

where $\varepsilon = \pm 1$.

Now we assume that $\phi(24\alpha^8 - 72\alpha^6(\alpha + \beta)^2) \equiv 0$; then we show that (4.1.24) still holds with $\lambda = 0$. Let $U = \{p \in M | \beta_p \not\equiv 0\}$. By assumption $\beta \not\equiv 0$, we know that U is a non-empty open subset of M . For a point $p \in U$, for any $y \in T_p M$ with $\beta_p(y) \neq 0$, we have

$$\psi_p(y) = 0.$$

Since the hyperplane $\beta_p = 0$ has zero measure in $T_p M$, we conclude that $\phi_p = 0$. Then β is parallel and $\psi = 0$ on U . It follows from (4.1.14) that the following holds on U .

$$\lambda(\alpha + \beta)^6 - \mu\alpha^6 = 0.$$

$$\mu = \lambda \frac{(\alpha + \beta)^6}{\alpha^6}.$$

We conclude that

$$\mu = 0.$$

According to Theorem (4.1.3(a)), we can state that: □

Theorem 4.1.4. *If $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ is locally projectively flat cubic (α, β) -metric, then F is locally Minkowskian.*

4.2 Geometric Equations for the Projectively Flat Generalized Matsumoto Metric

In this section, we examine locally projectively flat generalized Matsumoto metrics with constant Ricci curvature.

Let $F = \frac{\alpha^{p+1}}{(\alpha - \beta)^p}$ be a generalized Matsumoto metric on a manifold M , where

$$\alpha(y) = \sqrt{a_{ij}(x)y^i y^j}, \quad \beta(y) = b_i(x)y^i,$$

with $\|\beta\|_q = \sup_{y \in T_q M} \beta(y)/\alpha(y) < 1$.

For the generalized Matsumoto metric $F = \frac{\alpha^{p+1}}{(\alpha - \beta)^p}$,

$$F_1 = \frac{\alpha^p (\alpha - \beta(p+1))}{(\alpha - \beta)^{p+1}}, \quad F_2 = \frac{p\alpha^{p+1}}{(\alpha - \beta)^{p+1}},$$

$$F_{11} = \frac{p(p+1)\beta^2 \alpha^{p-1}}{(\alpha - \beta)^{p+2}}, \quad F_{22} = \frac{p(p+1)\alpha^{p+1}}{(\alpha - \beta)^{p+2}}.$$

Where $F_1 = \partial F / \partial \alpha$, $F_{11} = \partial F_1 / \partial \alpha$, $F_2 = \partial F / \partial \beta$, $F_{22} = \partial F_2 / \partial \beta$. Then

$$T = \frac{\alpha^{4p}}{(\alpha - \beta)^{4p+2}} \{ (\alpha - \beta)(\alpha - \beta(p+1)) + p(p+1)\gamma^2 \},$$

$$\xi = \frac{\gamma^2 p(p+1)r_{00} + 2(\alpha - \beta)p\alpha^2 s_0}{2 \{ (\alpha - \beta)(\alpha - \beta(p+1)) + p(p+1)\gamma^2 \}},$$

$$\eta = \frac{p\alpha^2 \{ r_{00}(\alpha - \beta(p+1)) - 2p\alpha^2 s_0 \}}{2 \{ (\alpha - \beta)(\alpha - \beta(p+1)) + p(p+1)(\Delta - \beta^2) \}}.$$

Substitute these in equation (4.1.3), we get D^i as follows

$$D^i = \frac{p \{ r_{00}(\alpha - \beta(p+1)) - 2p\alpha^2 s_0 \}}{2 \times \{ (\alpha - \beta)(\alpha - \beta(p+1)) + p(p+1)(\Delta - \beta^2) \}} \times \quad (4.2.1)$$

$$\left\{ y^i + \frac{\alpha^2(p+1)p^i}{(\alpha - \beta(p+1))} \right\} + \left(\frac{p\alpha^2}{(\alpha - \beta(p+1))} \right) s_0^i.$$

Where $p^i = B^i - \frac{\beta}{\alpha^2} y^i$, $\Delta = \gamma^2 + \beta^2$.

For Projectively flat generalized Matsumoto Metric, we have

$$s_0^i = \frac{\beta s_i + 2p^2(p+1)s_0 B^i b^2}{2pb^2(1 + p(p+1)b^2)}.$$

Then D^i can be written as

$$D^i = \frac{(A_1 r_{00} + A_2 s_0) y^i + (B_2 r_{00} + B_3 s_0) B^i + B_4 s^i}{B_5}, \quad (4.2.2)$$

where,

$$\begin{aligned} A_1 &= \frac{p\alpha^2(\alpha - \beta(p+1)) \times \{2pb^2(1 + p(p+1)b^2) \times \{(\alpha - \beta(p+1)) - \beta(p+1)\}\}}{2\{(\alpha - \beta)(\alpha - \beta(p+1)) + p(p+1)(b^2\alpha^2 - 2\beta^2)\}}, \\ A_2 &= \frac{-2p^2\alpha^4 \times \{2pb^2(1 + p(p+1)b^2) \times \{(\alpha - \beta(p+1)) - \beta(p+1)\}\}}{2\{(\alpha - \beta)(\alpha - \beta(p+1)) + p(p+1)(b^2\alpha^2 - 2\beta^2)\}}, \\ B_2 &= \frac{p\alpha^4 \{(\alpha - \beta(p+1)) \times 2pb^2(1 + p(p+1)b^2)(p+1)\}}{2\{(\alpha - \beta)(\alpha - \beta(p+1)) + p(p+1)(b^2\alpha^2 - 2\beta^2)\}}, \\ B_3 &= \frac{-2p^2\alpha^6 \times 2pb^2(1 + p(p+1)b^2)(p+1)}{2\{(\alpha - \beta)(\alpha - \beta(p+1)) + p(p+1)(b^2\alpha^2 - 2\beta^2)\}} + 2p^3(p+1)\alpha^4 b^2, \\ B_4 &= p\alpha^4 \beta, \end{aligned}$$

$$B_5 = \alpha^2(\alpha - \beta(p+1))2pb^2(1 + p(p+1)b^2),$$

$$B^i = a^{ir} b_r, \quad Y_i = a_{ir} y^r, \quad b^2 = a^{ij} b_i b_j, \quad s^i = a^{ir} s_r s^j y_j, \quad s_i = b_r s_i^r.$$

Lemma 4.2.1. For a projectively generalized Matsumoto metric $F = \frac{\alpha^{p+1}}{(\alpha - \beta)^p}$, D^i is given with the aid of equation (4.2.2).

If we define [10]

$$K_{jk}^i = \frac{a_{jk} s^i + r_{jk} B^i}{b^2},$$

then D^i are given by

$$D^i = \frac{(A_1 r_{00} + A_2 s_0) y^i + (B'_2 r_{00} + B'_3 k_{000}) B^i + B'_4 k_{00}^i}{B_5}, \quad (4.2.3)$$

where,

$$\begin{aligned} A_1 &= \frac{p\alpha^2(\alpha - \beta(p+1)) \times \{2pb^2(1 + p(p+1)b^2) \times \{(\alpha - \beta(p+1)) - \beta(p+1)\}\}}{2\{(\alpha - \beta)(\alpha - \beta(p+1)) + p(p+1)(b^2\alpha^2 - 2\beta^2)\}}, \\ A_2 &= \frac{-2p^2\alpha^4 \times \{2pb^2(1 + p(p+1)b^2) \times \{(\alpha - \beta(p+1)) - \beta(p+1)\}\}}{2\{(\alpha - \beta)(\alpha - \beta(p+1)) + p(p+1)(b^2\alpha^2 - 2\beta^2)\}}, \\ B'_2 &= \frac{p\alpha^4 \{(\alpha - \beta(p+1)) \times 2pb^2(1 + p(p+1)b^2)(p+1)\}}{2\{(\alpha - \beta)(\alpha - \beta(p+1)) + p(p+1)(b^2\alpha^2 - 2\beta^2)\}} \\ &\quad - \beta \left\{ \frac{-2p^2\alpha^4 \times 2pb^2(1 + p(p+1)b^2)(p+1)}{2\{(\alpha - \beta)(\alpha - \beta(p+1)) + p(p+1)(b^2\alpha^2 - 2\beta^2)\}} + 2p^3(p+1)\alpha^2 b^2 \right\} - p\alpha^2\beta, \\ B'_3 &= b^2 \left(\frac{-2p^2\alpha^4 \times 2pb^2(1 + p(p+1)b^2)(p+1)}{2\{(\alpha - \beta)(\alpha - \beta(p+1)) + p(p+1)(b^2\alpha^2 - 2\beta^2)\}} + 2p^3(p+1)\alpha^2 b^2 \right), \\ B'_4 &= p\alpha^2\beta b^2, \\ B_5 &= \alpha^2(\alpha - \beta(p+1))2pb^2(1 + p(p+1)b^2). \end{aligned}$$

Where $K_{ijk} = a_{jr} K_{ik}^r$.

Substituting (4.2.3) into G^i , we obtain

$$G^i = \bar{G}^i + \left\{ \frac{(A_1 r_{00} + A_2 s_0) + (B'_2 r_{i0} + B'_3 k_{i00}) B^i + B'_4 k_{i0}^i}{B_5} \right\} y^i, \quad (4.2.4)$$

Let

$$P = \left\{ \frac{(A_1 r_{00} + A_2 s_0) + (B'_2 r_{i0} + B'_3 k_{i00}) B^i + B'_4 k_{i0}^i}{B_5} \right\},$$

where all coefficients are as defined above. Then (4.2.4) becomes

$$G^i = \bar{G}^i + P y^i,$$

The S-curvature is given by

$$S = (n+1)\{P - dp(y)\}.$$

Substituting P into the above equation, we obtain

$$S = (n+1) \left\{ \frac{(A_1 r_{00} + A_2 s_0) + (B'_2 r_{i0} + B'_3 k_{i00}) B^i + B'_4 k_{i0}^i}{B_5} - dp(y) \right\}, \quad (4.2.5)$$

where $p := \ln \sqrt{1 - \|\beta\|^2}$.

Theorem 4.2.2. *The S -curvature of a projectively flat generalized Matsumoto metric is given by (4.2.5).*

4.2.1 Projectively Flat Generalized Matsumoto Metric with $K = 0$

Theorem 4.2.3. *Let $F = \frac{\alpha^{p+1}}{(\alpha-\beta)^p}$ be a locally projectively flat generalized Matsumoto metric. Assume that F has constant Ricci curvature $\text{Ric} = (n-1)\lambda F^2$, then:*

(a) F has flag curvature $K = \lambda = 0$,

(b) α has curvature $\tilde{K} = \mu = 0$.

Now from equation (4.1.13); we get

$$\frac{3}{4}\phi^2 = \frac{\lambda (\alpha^{4p+4})}{(\alpha-\beta)^{4p}} - \frac{\mu \alpha^{2p+4}}{(\alpha-\beta)^{2p}} + \frac{\psi (\alpha-\beta)^p (\alpha^{2p+2})}{\alpha^{p+1} (\alpha-\beta)^{2p}} - \frac{\psi (\alpha-\beta)^p (\alpha^{2p+2})}{2 (\alpha^{p+1}) (\alpha-\beta)^{2p}}. \quad (4.2.6)$$

Further, (4.2.6) is equivalent to

$$\frac{3}{4}\phi^2 = \frac{\lambda (\alpha^{4p+4})}{(\alpha-\beta)^{4p}} + \frac{\psi (\alpha-\beta)^p (\alpha^{2p+2})}{\alpha^{p+1} (\alpha-\beta)^{2p}}. \quad (4.2.7)$$

$$\frac{\psi}{2} = \frac{-\mu \alpha^{p+3}}{\alpha-\beta}. \quad (4.2.8)$$

Substituting (4.2.8) into (4.2.7), we get

$$\frac{3}{4}\phi^2 = \lambda \left(\frac{\alpha^{4p+4}}{(\alpha-\beta)^{4p}} \right) - \frac{2\mu \alpha^{3p+5}}{(\alpha-\beta)^{2p+2}}. \quad (4.2.9)$$

By differentiating equation (4.2.9) with respect to y^i and y^j , we obtain

$$\frac{3}{2}\phi b_{i|j|k}y^iy^j = \lambda \left[\frac{4p\alpha^{4p+4}(\alpha - \beta)^{4p-1}b_{i|k}y^i}{(\alpha - \beta)^{8p}} \right] - 2\mu \left[\frac{\alpha^{3p+5}(2p+2)(\alpha - \beta)^{2p+1}b_{i|k}y^i}{(\alpha - \beta)^{4p+4}} \right]. \quad (4.2.10)$$

Engaging in contracting (4.2.10) with y^k , we possess;

$$\frac{3}{2}\phi\psi = \frac{4p\lambda\alpha^{4p+4}(\alpha - \beta)^{4p-1}\phi}{(\alpha - \beta)^{8p}} - \frac{2\mu\phi\alpha^{3p+5}(2p+2)(\alpha - \beta)^{2p+1}}{(\alpha - \beta)^{4p+4}}. \quad (4.2.11)$$

By inserting (4.2.8) into (4.2.11) and we have;

$$\mu \left(\frac{-3\alpha^{p+3}}{\alpha - \beta} + \frac{2\alpha^{3p+5}(2p+2)}{(\alpha - \beta)^{2p+3}} \right) = \frac{4\lambda p\alpha^{4p+4}}{(\alpha - \beta)^{p+1}}. \quad (4.2.12)$$

Differentiating equation (4.2.12) with respect to y^i and y^j yields the same results.

$$\mu \left(\frac{3}{4} \frac{(\alpha - \beta)^{p-1}b_{i|k}}{\alpha^{1+3p}} + \frac{(4p+4)(p+2)\alpha^{1-p}b_{i|k}}{4(\alpha - \beta)^{p+3}} \right) = 0. \quad (4.2.13)$$

Now, by once more contracting by y_k , we obtain:

$$\mu\phi \left(\frac{3}{4} \frac{(\alpha - \beta)^{p-1}}{\alpha^{1+3p}} + \frac{(4p+4)(p+2)\alpha^{1-p}}{4(\alpha - \beta)^{p+3}} \right) = 0. \quad (4.2.14)$$

Assume that;

$$\phi \left(\frac{3}{4} \frac{(\alpha - \beta)^{p-1}}{\alpha^{1+3p}} + \frac{(4p+4)(p+2)\alpha^{1-p}}{4(\alpha - \beta)^{p+3}} \right) \neq 0.$$

Implies that ;

$$\mu = 0.$$

By inserting it into (4.2.9), that results in;

$$\phi^2 = \frac{4\lambda}{3} \left(\frac{\alpha^{4p+4}}{(\alpha - \beta)^{4p}} \right). \quad (4.2.15)$$

We observe that: $\lambda \geq 0$ and;

$$\phi = \frac{2\varepsilon\sqrt{\lambda}}{\sqrt{3}} \frac{\alpha^{2p+2}}{(\alpha - \beta)^{2p}}, \quad (4.2.16)$$

where $\varepsilon = \pm 1$.

Assuming that $\phi \left(\frac{3}{4} \frac{(\alpha - \beta)^{p-1}}{\alpha^{1+3p}} + \frac{(4p+4)(p+2)\alpha^{1-p}}{4(\alpha - \beta)^{p+3}} \right) \equiv 0$, we will demonstrate that (4.2.16) remains valid with $\lambda = 0$. Let us define $U = \{q \in M \mid \beta_q \neq 0\}$. Given that $\beta \neq 0$, it follows that U is a non-empty open subset of M . For any point $q \in U$, and for every vector $y \in T_q M$ such that $\beta_q(y) \neq 0$, we obtain

$$\psi_q(y) = 0.$$

Since the hyperplane $\beta_q = 0$ has zero measure in $T_q M$, we conclude that $\phi_q = 0$. Consequently, β is parallel and $\psi = 0$ on U . From (4.2.6), the following holds on U :

$$\frac{\lambda(\alpha^{4p+4})}{(\alpha - \beta)^{4p}} = \frac{\mu\alpha^{2p+4}}{(\alpha - \beta)^{2p}}.$$

$$\mu = \lambda \frac{\alpha^{2p}}{(\alpha - \beta)^{2p}}.$$

We conclude that

$$\mu = 0.$$

From Theorem (4.2.3)(a), we have the following:

Theorem 4.2.4. *If $F = \frac{\alpha^{p+1}}{(\alpha - \beta)^p}$ is a locally projectively flat generalized Matsumoto metric, then F is locally Minkowskian.*

CHAPTER 5

Locally Dually Flatness Properties in Finsler

(α, β) -Metrics

In this chapter, we establish the criteria for a non-Riemannian cubic (α, β) -metric and an exponential (α, β) -metric to be locally dually flat and projectively flat and characterize these conditions in terms of isotropic S -curvature.

5.1 Introduction

Information geometry is a branch of geometry in which the concepts of differential geometry are applied to study the geometrical structure of probability distribution functions in the fields of control system theory, statistical inference and multi-terminal information theory [8, 9]. Amari and Nagaoka [9] considered the Riemannian metric and studied its dually flat properties in the context of information geometry. Further, Yu [162] proved that the spray coefficients for dual flatness of a Riemannian metric can be expressed as follows:

$$G_{\alpha}^i = 2\theta y^i + \alpha^2 \theta^i,$$

where G_{α}^i are the spray coefficients, $\theta := \theta_i(x)y^i$ is a 1-form, and $\theta^i := a^{ij}\theta_j$.

Thus, for the Riemannian metric, the characterization for dually flat is clear. However, in the case of Finsler metrics, it is more difficult yet interesting, as it involves the study of non-Riemannian spaces. The renowned mathematician Shen [125] applied this concept in the non-Riemannian case in a particular and simple class of Finsler metrics known as the Funk metric, which is given by

$$F = \frac{\sqrt{(1 - |x|^2|y|^2) + \langle x, y \rangle^2}}{1 - |x|^2} \pm \frac{\langle x, y \rangle}{1 - |x|^2},$$

and defined on the unit ball $B^n(\mu) \subseteq \mathbb{R}^n$. The above metric is the only known example of a locally dually flat metric with non-zero constant flag curvature $K = -\frac{1}{4}$, under which it is a special class of Randers metrics, making it a particularly interesting example in the case of non-Riemannian metrics. Furthermore, Cheng et al. [37] proved that the dual flatness of a Randers metric always arises from that of some Riemannian metric. In recent years, the concept of dually flat Finsler metrics has become an important class in Finsler geometry, and many authors [31, 32, 35, 36, 92, 93, 125, 152, 159] have studied dually flat Finsler information structures and obtained various interesting results that application in Information geometry.

5.1.1 Preliminaries

Definition 5.1.1. A Finsler metric F on an n -dimensional manifold M is said to have isotropic S -curvature if $S = (n+1)c(x)F$, for some scalar function c on M [122].

The mean Cartan torsion for an (α, β) -metric is given by [118]

$$I_i = -\frac{\Phi(\phi - s\phi')}{2\Delta\phi\alpha^2}(\alpha b_i - sy_i).$$

Thus, for an (α, β) -metric $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, we have:

Theorem 5.1.2. [158] If F is not Riemannian and $\phi'(s) \neq 0$, then F is locally dually flat on M if and only if the pair (α, β) and $\phi = \phi(s)$ satisfy:

1. $s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l)$;
2. $r_{00} = \frac{2}{3}\theta\beta + \left[\tau + \frac{2}{3}(b^2\tau - \theta_l b^l)\right]\alpha^2 + \frac{1}{3}(3k_2 - 2 - 3k_3 b^2)\tau\beta^2$;
3. $G_\alpha^l = \frac{1}{3}[2\theta + (3k_1 - 2)\tau\beta]y^l + \frac{1}{3}(\theta^l\tau b^l)\alpha^2 + \frac{1}{2}k_3\tau\beta^2 b^l$;
4. $\tau \left[s(k_2 - k_3 s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi') \right] = 0$.

where $\tau = \tau(x)$ is a scalar function, $\theta = \theta_i(x)y^i$ is a 1-form on M , $\theta^l = a^{lm}\theta_m$, $k_1 = \Pi(0)$, $k_2 = \frac{\Pi'(0)}{Q(0)}$, $k_3 = \frac{1}{6Q^2(0)} \left[3Q''(0)\Pi'(0) - 6\Pi^2(0) - Q(0)\Pi'''(0) \right]$ and $Q = \frac{\phi'}{(\phi - s\phi')}$, $\Pi = \frac{(\phi'^2 + \phi\phi'')}{(\phi(\phi - s\phi'))}$. From theorem (5.1.2), we have:

Lemma 5.1.3. ([36]) Let $F = F(x, y)$ be a Finsler metric on an open subset $U \subset R^n$. Then F is locally dually flat and projectively flat on U if and only if $F_{x^k} = CF_{y^k}$, where C is a constant.

Corollary 5.1.4. [158] Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be an (α, β) -metric on an n -dimensional manifold $M^n (n \geq 3)$, where $\alpha = \sqrt{(a_{ij}y^i y^j)}$ is a Riemannian metric and $\beta = b_i(x)y^i \neq 0$, is a 1-form on M . Suppose that F is not Riemannian and $\phi'(s) \neq 0$. If ϕ further satisfies

$$s(k_2 - k_3 s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi') \neq 0.$$

then F is locally dually flat on M if and only if α and β satisfy:

1. $s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l)$,
2. $r_{00} = \frac{2}{3}[\theta\beta - (\theta_l b^l)\alpha^2]$,
3. $G_\alpha^l = \frac{1}{3}[2\theta y^l + \theta^l \alpha^2]$.

Chen, Shen, and Zhou [36] explored the class of (α, β) -metrics of non-Randers type, i.e., $\phi \neq t_1\sqrt{1+t_2s^2} + t_3s$ with isotropic S -curvature, and Bacsó, Cheng, and Shen [14] established the following results:

Theorem 5.1.5. Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be a non-Riemannian (α, β) -metric on a manifold M and $\phi \neq t_1\sqrt{1+t_2s^2} + t_3s$ for any constants $t_1 > 0$, t_2 and t_3 then F is of isotropic S -curvature $S = (n+1)cF$ if and only if one of the following assertions holds: [118]

(i) β satisfies

$$r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), s_j = 0, \quad (5.1.1)$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, and $c = c(x)$ satisfies

$$\Phi = -2(n+1)k \frac{\phi\Delta^2}{(b^2 - s^2)}, \quad (5.1.2)$$

where k is a real constant. In this case, $S = (n+1)cF$ with $c = k\varepsilon$;

(ii) β satisfies

$$r_{ij} = 0, s_{ij} = 0. \quad (5.1.3)$$

In this scenario, $S = 0$ irrespective of the choice of a specific ϕ .

5.1.2 Characterization of Locally Dually Flat Cubic (α, β) - Metric

In the present section, we study locally dually flat cubic (α, β) metric given by equation (1.4.5) with isotropic S - curvature and obtained following results :

Theorem 5.1.6. *The cubic (α, β) -metric on an n -dimensional manifold M ($n \geq 3$) is locally dually flat if and only if the following conditions are satisfied simultaneously.*

1. $s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l)$;
2. $r_{00} = \frac{2}{3}\theta\beta + \left[\tau + \frac{2}{3}(b^2\tau - \theta_l b^l)\right]\alpha^2 + \frac{1}{3}(13 + 90b^2)\tau\beta^2$;
3. $G_\alpha^l = \frac{1}{3}[2\theta + 43\tau\beta]y^l + \frac{1}{3}(\theta^l\tau b^l)\alpha^2 - 15\tau\beta^2 b^l$.

Proof. For the Finsler metric $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ we can obtain $k_1 = 15, k_2 = 5, k_3 = -30$, and

$$\left\{ \begin{array}{l} \phi = (1+s)^3, \phi' = 3(1+s)^2, \phi'' = 6(1+s), \phi''' = 6, \Pi = \frac{15}{1-s-2s^2}, \\ \Pi(0) = 15, \Pi'(0) = 15, \Pi''(0) = 90, \Pi'''(0) = 450, \\ Q = \frac{3}{1-2s}, Q' = \frac{6}{(1-2s)^2}, Q'' = \frac{24}{(1-2s)^3}, Q''' = \frac{144}{(1-2s)^4} \\ Q(0) = 3, Q'(0) = 6, Q''(0) = 24, Q'''(0) = 144. \end{array} \right.$$

In view of the above values, Theorem (5.1.2) yields the following result.

$$\left[s(k_2 - k_3 s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi') \right] \neq 0 \text{ and } \tau = 0.$$

Finally, substituting k_1, k_2 , and k_3 into Theorem (5.1.2), the result follows immediately. $\square$

An immediate consequence of Corollary (5.1.4) is the following:

1. $s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l),$
2. $r_{00} = \frac{2}{3}[\theta\beta - (\theta_l b^l)\alpha^2],$
3. $G_\alpha^l = \frac{1}{3}[2\theta y^l + \theta^l \alpha^2].$

Thus, F is locally dually flat for the cubic (α, β) -metric.

Let $\phi = \phi(s)$ denote a positive C^∞ function on $(-b_0, b_0)$. Consider a number $b \in [0, b_0]$ such that

$$\Phi = -(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q'', \quad (5.1.4)$$

where $\Delta = 1 + sQ + (b^2 - s^2)Q'$. Now using the value of $Q = \frac{3}{1-2s}$ and $Q' = \frac{6}{(1-2s)^2}$ into the above equation, we obtain

$$\Delta = \frac{1 - s - 8s^2 + 6b^2}{(1 - 2s)^2}.$$

Eliminating $1 + sQ$ from the expression $\Delta = 1 + sQ + (b^2 - s^2)Q'$ and equation (5.1.4), we obtain

$$\Phi = -(Q - sQ')(n+1)\Delta + (b^2 - s^2)\{(Q - sQ')Q' - (1 + sQ)Q''\}. \quad (5.1.5)$$

Theorem 5.1.7. *Let F be a non-Randers cubic (α, β) -metric on an n -dimensional manifold M ($n \geq 3$) with isotropic S -curvature $S = (n+1)cF$, where $c = c(x)$ is a scalar function. If F is locally dually flat, then F is locally projectively flat, and its spray coefficients satisfy $G^i = 0$.*

Proof. Let $G^i = G^i(x, y)$ and $\bar{G}_\alpha^i = \bar{G}_\alpha^i(x, y)$ denote the coefficients of F and α , respectively, with respect to the same coordinate system. By definition [118], we have

$$G^i = \bar{G}_\alpha^i + Py^i + Q^i, \quad (5.1.6)$$

where

$$P = \alpha^{-1}\Theta - 2Q\alpha_{s00} + r_{00}, \quad (5.1.7)$$

$$Q^i = \alpha Q s_0^i + \Psi - 2Q\alpha_{s00} + r_{00}b^i, \quad (5.1.8)$$

$$\begin{cases} \Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')} = \frac{3 - 12s}{2(1 - s - 8s^2 + 6b^2)}, \\ \Psi = \frac{1}{2} \frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''} = \frac{3}{1 - s - 8s^2 + 6b^2}. \end{cases}$$

First, we assume that Theorem (5.1.5) holds in case (i). For the cubic (α, β) -metric, we have

$$\Delta = \frac{1 - s - 8s^2 + 6b^2}{(1 - 2s)^2}.$$

From the above expression it is clear that $(1 - 2s)^2\Delta$ is a polynomial of degree 2 with respect to s . On the other hand, we have

$$\phi\Delta^2 = \frac{\phi(1 - s - 8s^2 + 6b^2)^2}{(1 - 2s)^4}. \quad (5.1.9)$$

So, if case (ii) of Theorem (5.1.5) holds, then replacing the aforementioned value of $\phi\Delta^2$ in equation (5.1.2), we have

$$(1 - 2s)^4(b^2 - s^2)\Phi = -2(n + 1)k\phi(1 - s - 8s^2 + 6b^2)^2. \quad (5.1.10)$$

It follows that $(b^2 - s^2)(1 - 2s)^4\Phi$ is not a polynomial in s . Thus, equation (5.1.9) can be rewritten as

$$\phi\Delta^2 = \frac{\bar{\Delta}}{(1 - 2s)^4},$$

where

$$\bar{\Delta} = \phi(1 - s - 8s^2 + 6b^2)^2.$$

Since the metric F is of non-Randers type and $\bar{\Delta}$ is not a polynomial in s , then $(b^2 - s^2)(1 - 2s)^4\Phi$ is not a polynomial in s . Now, using the value of $Q = \frac{3}{1 - 2s}$, $Q' = \frac{6}{(1 - 2s)^2}Q'' =$

$\frac{24}{(1-2s)^3}$ in equation (5.1.5), we have

$$\Phi = \frac{-(n+1)(3-12s)(1-s-8s^2+6b^2) + (b^2-s^2)(-6-96s)}{(1-2s)^4}. \quad (5.1.11)$$

From equations (5.1.10) and (5.1.11), the expressions $(b^2-s^2)(1-2s)^4\Phi$ are polynomials of degree 4 and 5 with respect to s , respectively. From equation (5.1.11) the coefficient of s^5 is never zero ensuring that $\Phi \neq 0$. So, we conclude that equation (5.1.2) is not viable. Thus, case (ii) of Theorem (5.1.5) holds. Hence, we have

$$r_{00} = 0, s_{ij} = 0.$$

Using $r_{00} = 0$ in the second condition of Theorem (5.1.6), we have

$$\left[\tau + \frac{2}{3}(b^2\tau - \theta_l b^l) \right] \alpha^2 = \frac{1}{3}\beta [-2\theta - (13+90b^2)\beta\tau]. \quad (5.1.12)$$

While α^2 is an irreducible polynomial in y^i , equation (5.1.12) yields

$$\tau + \frac{2}{3}(b^2\tau - b_m \theta^m) = 0, \quad (5.1.13)$$

$$\frac{2}{3}\theta + \frac{1}{3}(13+90b^2)\beta\tau = 0, \quad (5.1.14)$$

From equation (5.1.14), we obtain

$$\theta = -\frac{1}{2}(13+90b^2)\beta\tau. \quad (5.1.15)$$

Contracting the first condition of Theorem (5.1.6) by y^l , we have

$$s_{00} = -\frac{1}{3}(\theta b^2 - \beta b_l \theta^l).$$

Using the condition $s_{00} = 0$ in the above equation, we have

$$(\theta b^2 - \beta b_l \theta^l) = 0. \quad (5.1.16)$$

Using equations (5.1.15) and (5.1.16) in equation (5.1.13), we have

$$\tau = 0$$

Now substituting the above value of $\tau = 0$ in equation (5.1.15), we get

$$\theta = 0.$$

Thus the first, second, and third conditions of Theorem (5.1.6) reduce to

$$s_{ij} = 0, \quad G_\alpha^l = 0, \quad r_{00} = 0. \quad (5.1.17)$$

As $s_{00} = r_{00} = 0$, equations (5.1.7) and (5.1.8) simplify to

$$P = 0 \quad \text{and} \quad Q^i = 0. \quad (5.1.18)$$

Thus, using equations (5.1.17) and (5.1.18) in equation (5.1.6), we get

$$G^i = 0.$$

This completes the proof. □

Theorem 5.1.8. *Consider a non-Riemannian cubic (α, β) -metric F on an n -dimensional manifold M , where $n \geq 3$. If F is locally dually flat and admits isotropic S -curvature $S = (n+1)cF$, where $c = c(x)$ is a scalar function, then F is locally Minkowskian.*

Proof. From Theorems (5.1.6) and (5.1.7), it follows that the cubic (α, β) -metric is dually flat and projectively flat in any adapted coordinate system, respectively. Furthermore, by

Lemma (5.1.3), we deduce that

$$F_{x^k} = CF_{y^k}.$$

The spray coefficients $G^i = Py^i$ can be expressed as $P = \frac{1}{2}CF$. By Theorem (5.1.7), if $G^i = 0$, then $P = 0$, and hence $C = 0$. This means that $F_{x^k} = 0$. Thus, the cubic (α, β) -metric is a locally Minkowskian metric in the adapted coordinate system. $\square$

5.2 Characterization of Locally Dually Flat Exponential (α, β) -Metric

In this section, we investigate the exponential (α, β) -metric given by (1.4.6), which is locally dually flat and possesses isotropic S -curvature. We present the following results.

Theorem 5.2.1. *The necessary and sufficient conditions for an exponential (α, β) -metric to be locally dually flat on an n -dimensional manifold ($n \geq 3$) are as follows:*

1. $s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l)$;
2. $r_{00} = \frac{2}{3}\theta\beta + \left[\tau + \frac{2}{3}(b^2\tau - \theta_l b^l)\right]\alpha^2 + \frac{1}{12}(-56 - 159b^2)\tau\beta^2$;
3. $G_\alpha^l = \frac{1}{3}(2\theta + 7\tau\beta)y^l + \frac{1}{3}(\theta^l\tau b^l)\alpha^2 + \frac{53}{8}\tau\beta^2 b^l$.

Proof. For the Finsler metric $F = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{\frac{-\beta}{\alpha}}$, we obtain $k_1 = 3, k_2 = -4, k_3 = \frac{53}{4}$ and

$$\begin{aligned}\phi &= e^s + se^{-s}, \\ \phi' &= e^s + e^{-s} - se^{-s}, \\ \phi'' &= e^s - 2e^{-s} + se^{-s}, \\ \phi''' &= e^s + 3e^{-s} - se^{-s},\end{aligned}$$

$$\Pi = \frac{2e^{2s} + e^{-2s} - 4se^{-2s} + 2s^2e^{-2s}}{e^{2s} - se^{2s} + s + s^3e^{-2s}},$$

$$\Pi(0) = 3, \quad \Pi'(0) = -8, \quad \Pi''(0) = 64, \quad \Pi'''(0) = -210.$$

$$Q = \frac{e^s + e^{-s} - se^{-s}}{e^s - se^s + s^2e^{-s}},$$

$$Q' = \frac{(e^s - 2e^{-s} + se^{-s})(e^s + se^{-s})}{(e^s - se^s + s^2e^{-s})^2},$$

$$Q'' = \frac{N_1 + N_2}{(e^s - se^s + s^2e^{-s})^3},$$

$$Q''' = \frac{A}{B},$$

where,

$$N_1 = (e^s - se^s + s^2e^{-s})(2e^{2s} + 2 - 2e^{-2s} + 6se^{-2s} - 2s^2e^{-2s}),$$

$$N_2 = 2s(e^s - 2e^{-s} + se^{-s})(e^{2s} + 2s - 2 - 2se^{-2s} + s^2e^{-2s}),$$

and,

$$\begin{aligned} A = & (6e^{3s} - 8e^s + 18e^{-s} - 60se^{-s} - 6se^s + 46s^2e^{-s} - 8s^3e^{-s} + 8s^2e^s \\ & + 12se^{-3s} - 24s^2e^{-3s} + 6s^3e^{-3s}) \times \\ & - 3(-se^s + 2se^{-s} - s^2e^{-s})(2e^{3s} + 2e^s - 2e^{-s} + 16se^{-s} - 10se^s \\ & - 22s^2e^{-s} + 8s^3e^{-s} + 8s^2e^s + 6s^2e^{-3s} + 2s^3e^{-3s} - 4s^3e^{-3s}), \end{aligned}$$

$$B = (e^s - se^s + s^2e^{-s})^4.$$

$$Q(0) = 2, \quad Q'(0) = -1, \quad Q''(0) = 2, \quad Q'''(0) = 10.$$

Utilizing the aforementioned values, Theorem (5.1.2) yields

$$\left[s(k_2 - k_3s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi') \right] \neq 0 \text{ and } \tau = 0.$$

To conclude the claim, we replace k_1, k_2 , and k_3 in Theorem (5.1.2). □

Corollary (5.1.4) states that the exponential (α, β) -metric satisfies the criterion

$$s(k_2 - k_3 s^2)(\phi\phi' - s(\phi')^2 - s\phi\phi'') - ((\phi')^2 + \phi\phi'') + k_1\phi(\phi - s\phi') \neq 0.$$

We have

1. $s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l),$
2. $r_{00} = \frac{2}{3}\theta\beta - \frac{2}{3}\theta_l b^l \alpha^2,$
3. $G_\alpha^l = \frac{1}{3}(2\theta)y^l.$

Thus, F is locally dually flat for the exponential (α, β) -metric. Now, let $\phi = \phi(s)$ be a positive C^∞ function defined on $(-b_0, b_0)$. For any $b \in [0, b_0]$, let

$$\Phi = -(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q'', \quad (5.2.1)$$

where $\Delta = 1 + sQ + (b^2 - s^2)Q'$. Now, using the values of $Q = \frac{e^s + e^{-s} - se^{-s}}{e^s - se^s + s^2e^{-s}}$ and $Q' = \frac{(e^s - 2e^{-s} + se^{-s})(e^s + se^{-s})}{(e^s - se^s + s^2e^{-s})^2}$ in the above equation, we have

$$\Delta = \frac{k_1}{k_2}.$$

Where,

$$k_1 = (e^s - se^s + s^2e^{-s})^2 + s(e^s + e^{-s} - se^{-s}) \times (e^s - se^s + s^2e^{-s}) + (b^2 - s^2)(e^{2s} + 2s - 2 - 2se^{-2s} + s^2e^{-2s}), \quad k_2 = (e^s - se^s + s^2e^{-s})^2.$$

Eliminating $1 + sQ$ from $\Delta = 1 + sQ + (b^2 - s^2)Q'$ and equation (5.2.1), we have

$$\Phi = -(Q - sQ')(n + 1)\Delta + (b^2 - s^2)\{(Q - sQ')Q' - (1 + sQ)Q''\}. \quad (5.2.2)$$

Theorem 5.2.2. *If F non-Randers type exponential (α, β) -metric on n -dimensional manifold M ($n \geq 3$) and locally dually flat with isotropic S curvatures $S = (n + 1)cF$, then F is locally projectively flat in adapted coordinate system and $G^i = 0$.*

Proof. Let $G^i = G^i(x, y)$ and $\bar{G}_\alpha^i = \bar{G}_\alpha^i(x, y)$ denote the spray coefficients of F and α , respectively, with respect to the same coordinate system. By definition [118], we have

$$G^i = \bar{G}_\alpha^i + Py^i + Q^i, \quad (5.2.3)$$

where

$$P = \alpha^{-1}\Theta - 2Q\alpha s_{00} + r_{00}, \quad (5.2.4)$$

$$Q^i = \alpha Q s_0^i + \Psi - 2Q\alpha s_{00} + r_{00}b^i, \quad (5.2.5)$$

$$\left\{ \begin{aligned} \Theta &= \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')} \\ &= \frac{e^{2s} + 1 + 3s^2e^{-2s} - 2se^{2s} - 2s^3e^{-2s}}{2(e^s + se^{-s})\{(e^s - se^s + s^2e^{-s}) + (b^2 - s^2)(e^s - 2e^{-s} + se^{-s})\}}, \end{aligned} \right.$$

$$\left\{ \begin{aligned} \Psi &= \frac{1}{2} \frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''} \\ &= \frac{1}{2} \left\{ \frac{e^s - 2e^{-s} + se^{-s}}{(e^s - se^s + s^2e^{-s}) + (b^2 - s^2)(e^s - 2e^{-s} + se^{-s})} \right\}. \end{aligned} \right.$$

Initially, we assume that case (i) of Theorem (5.1.5) holds. Remarkably, using the exponential (α, β) -metric, we have

$$\Delta = \frac{r_1}{r_2}.$$

Where,

$$r_1 = (1 - s + s^2e^{-2s}) + s(1 + e^{-2s} - se^{-2s})(1 - s + s^2e^{-2s}) + (b^2 - s^2)(1 + 2se^{-2s} - 2e^{-2s} - 2se^{-4s} + s^2e^{-4s}), \quad r_2 = (1 - s + s^2e^{-2s})^2.$$

Expanding the exponential (α, β) -metric in series form and choosing the first term, we obtain that $(1 - s + s^2)^2\Delta$ is a polynomial of degree four with respect to s . On the other hand, we have

$$\phi\Delta^2 = \frac{\phi(1 + s - s^2 + 3s^3 - 2s^4 + b^2(s^2 - 1))^2}{(1 - s + s^2)^4}. \quad (5.2.6)$$

If case (ii) of Theorem (5.1.5) applies, then substituting the aforementioned value of $\phi\Delta^2$

in equation (5.1.2), we have

$$(1 - s + s^2)^4(b^2 - s^2)\Phi = -2(n+1)k\phi(1 + s - s^2 + 3s^3 - 2s^4 + b^2(s^2 - 1))^2. \quad (5.2.7)$$

It implies that $(1 - s + s^2)^4(b^2 - s^2)\Phi$ is not a polynomial in s (if $k = 0$, the Cartan torsion equation leads to a contradiction). Hence, equation (5.2.6) can be rewritten as:

$$\phi\Delta^2 = \frac{\bar{\Delta}}{(1 - s + s^2)^4},$$

where

$$\bar{\Delta} = \phi(1 + s - s^2 + 3s^3 - 2s^4 + b^2(s^2 - 1))^2.$$

Since the metric F is a non-Randers types and $\bar{\Delta}$ is not a polynomial in s , then $(1 - s + s^2)^4(b^2 - s^2)\Phi$ is not a polynomial in s . Now, using the value of

$$\begin{aligned} Q &= \frac{e^s + e^{-s} - se^{-s}}{e^s - se^s + s^2e^{-s}}, \quad Q' = \frac{(e^s - 2e^{-s} + se^{-s})(e^s + se^{-s})}{(e^s - se^s + s^2e^{-s})^2}, \\ Q'' &= \frac{(e^s - se^s + s^2e^{-s})(2e^{2s} + 2 - 2e^{-2s} + 6se^{-2s} - 2s^2e^{-2s})}{(e^s - se^s + s^2e^{-s})^3} \\ &\quad + \frac{2s(e^s - 2e^{-s} + se^{-s})(e^{2s} + 2s - 2 - 2se^{-2s} + s^2e^{-2s})}{(e^s - se^s + s^2e^{-s})^3}, \end{aligned}$$

in equation (5.2.2), we have

$$\Phi = \frac{-(n+1)(1 + s - s^2 + 3s^3 - 2s^4 + b^2(s^2 - 1) - s(s^2 - 1)) \times (1 - s + s^2)^2}{(1 - s + s^2)^4}. \quad (5.2.8)$$

According to equations (5.2.7) and (5.2.8), the expression $(b^2 - s^2)(1 - 2s)^4\Phi$ is a polynomial of degree 8 and 11 with respect to s , respectively. From equation (5.2.8), the coefficient of s^{11} is never zero, hence $\Phi \neq 0$. Therefore, we conclude that equation (5.1.2) is not viable. Thus, case (ii) of Theorem (5.1.5) holds. So, we have

$$r_{00} = 0, s_{ij} = 0.$$

Using $r_{00} = 0$ in the second condition of Theorem (5.2.1), we have

$$\left[\tau + \frac{2}{3}(b^2\tau - \theta_l b^l) \right] \alpha^2 = \frac{-1}{3} \beta \left[-2\theta + \left(\frac{56 + 159b^2}{4} \right) \beta \tau \right]. \quad (5.2.9)$$

Although α^2 is an irreducible polynomial in y^i , equation (5.2.9) yields

$$\tau + \frac{2}{3}(b^2\tau - b_m \theta^m) = 0, \quad (5.2.10)$$

$$\frac{-2}{3} \theta + \left(\frac{56 + 159b^2}{4} \right) \beta \tau = 0. \quad (5.2.11)$$

From equation (5.2.11), we obtain the value of θ as follows:

$$\theta = \frac{1}{2} \left(\frac{56 + 159b^2}{4} \right) \beta \tau. \quad (5.2.12)$$

Contracting the first condition of Theorem (5.2.1) by y^l , we have

$$s_{00} = -\frac{1}{3}(\theta b^2 - \beta b_l \theta^l).$$

Assuming $s_{00} = 0$ in the above equation, we obtain

$$(\theta b^2 - \beta b_l \theta^l) = 0. \quad (5.2.13)$$

Substituting equations (5.2.12) and (5.2.13) into equation (5.2.10), we get

$$\tau = 0.$$

Substituting the above value of $\tau = 0$ into equation (5.2.12), we get

$$\theta = 0.$$

The first, second, and third conditions of Theorem (5.2.1) reduce to

$$s_{ij} = 0, \quad G_\alpha^l = 0, \quad r_{00} = 0. \quad (5.2.14)$$

Since $s_{00} = r_{00} = 0$, equations (5.2.4) and (5.2.5) simplify to

$$P = 0 \quad \text{and} \quad Q^i = 0. \quad (5.2.15)$$

By applying equations (5.2.14) and (5.2.15) to equation (5.2.3), we obtain

$$G^i = 0.$$

This completes the proof. □

Theorem 5.2.3. *If F is a non-Riemannian exponential (α, β) -metric on an n -dimensional manifold M ($n \geq 3$) and locally dually flat with isotropic S -curvature, then $S = (n + 1)cF$ if and only if F is locally Minkowskian.*

Proof. The exponential (α, β) -metric is dually flat and projectively flat in any adapted coordinate system, as shown in Theorems (5.2.1) and (5.2.2). Based on Lemma (5.1.3), we deduce

$$F_{x^k} = C F F_{y^k}.$$

By Theorem (5.2.2), $G^i = 0 \Rightarrow C = 0$, i.e., $F_{x^k} = 0$, thus the exponential (α, β) -metric is locally Minkowskian in the adapted coordinate system. □

CHAPTER 6

A New Class of Finsler Metrics

This chapter demonstrates how to construct a new type of Finsler metric that is not always an (α, β) -metric. The metric mentioned above is determined by a Riemannian metric and two 1-forms, $\beta = b_i(x)y^i$ and $\gamma = d_i(x)y^i$. We also obtain a variety of significant geometric results when employing this type of Finsler metric.

6.1 Introduction

Chern [40] described Finsler geometry as an extension of Riemannian geometry that relaxes the restriction of quadratic dependence on directional variables. This generalization offers greater flexibility in defining both distance and curvature, thereby establishing Finsler geometry as a powerful and versatile framework within differential geometry. This capacity to model spaces where geometric properties vary with direction has rendered it an increasingly valuable tool in both theoretical and applied fields. Spanning fields from general relativity to biology and information theory, Finsler geometry provides a unified approach to describing anisotropic and direction-dependent phenomena. Although the field has seen significant advancements in recent decades, many foundational questions remain unresolved and continue to attract active research. This ongoing development underscores the richness of the field and its vast potential for future exploration.

In 2021, Rajabi and Sadeghzadeh [106] proposed a novel class of Finsler metrics, which are characterized by a Finsler metric F and a 1-form metric $\gamma = \gamma_i y^i$ on a n -dimensional manifold. This new class is defined by the functional form $\bar{F} = F\psi(\bar{s})$, where $\bar{s} = \frac{\gamma}{F}$, $\|\gamma\|_F < g_0$ and $\psi(\bar{s})$ represents a positive C^∞ function on $(-g_0, g_0)$ [118]. In their research, the authors investigated the Cartan connection associated with these metrics, established projectivity criteria, and derived several new classes of Douglas metrics from this class.

The new Finsler metrics are expressed in the following form:

$$\tilde{F} = F\phi(s), \quad (6.1.1)$$

where $s = \frac{\beta}{F}$, $\beta = b_i(x)y^i$, $\|\beta\|_F < b_0$, $\phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$, and F is a Finsler metric. These metrics, which we call (F, β) -metrics, are not merely restricted to (α, β) -metrics, even when F is itself an (α, β) -metric; they also broaden and extend the overall class of Finsler metrics.

The concept of an h -vector b_i , which depends on the coordinates x^i and the directional variables y^i , was first presented by Izumi [60] in 1980. It satisfies the following condition:

$$FC_{ij}^h b_h = \rho h_{ij},$$

where C_{ij}^h are components of the Cartan tensor. The components of the angular metric tensor are denoted by h_{ij} , and ρ denotes a non-zero scalar function. By establishing a fundamental relationship among the angular metric tensor, the h -vector, and the Cartan connection, this equation yields significant insight into the conformal structure of Finsler spaces.

In the present work, we investigate a new class of Finsler metrics defined by the expression:

$$\tilde{F} = \frac{F^2}{F + \beta} = F \frac{1}{1 + \frac{\beta}{F}} = F\phi(s);$$

where $\phi(s) = \frac{1}{1+s}$, $s = \frac{\beta}{F}$. To explore the geometric significance of this metric, we consider the special case where $F = \alpha + \gamma$ is a Randers metric, with α a Riemannian metric and $\gamma = d_i(x)y^i$ is a 1-form on the manifold M the expression for $\tilde{F}s$ becomes:

$$\tilde{F} = \frac{(\alpha + \gamma)^2}{\alpha + \gamma + \beta} = \alpha \frac{(1 + \bar{s})^2}{1 + \bar{s} + s}$$

where $s = \frac{\beta}{\alpha}$ and $\bar{s} = \frac{\gamma}{\alpha} \neq s$.

Hence, $\tilde{F} = \alpha\psi(s, \bar{s})$ defines a Finsler metric that does not belong to the standard (α, β) -

type. In contrast, if one adds a 1-form to the metric, as in $\tilde{F} = F + \beta = \alpha + \beta + \gamma$, the result is a Randers metric. This distinction highlights that the new class of metrics introduced here generally lies outside the realm of (α, β) -metrics, even though the original metric F may belong to that class. In this work, we focus on the metric $\tilde{F} = F\psi(s)$, constructed by means of an h -vector, and establish conditions under which this metric becomes projectively flat (Theorem 6.1.1). Furthermore, in Theorem (6.1.3), we derive the conditions under which $\tilde{F}$ becomes a Douglas metric. The main results of this work are summarized as follows:

Theorem 6.1.1. *An (F, β) -metric of the form $\tilde{F} = (\alpha + \gamma)\phi(s)$, is projectively flat if and only if the following condition is satisfied:*

$$\left\{ \begin{array}{l} 2 \left\{ \frac{1+2s-\rho}{(1+s)^2} \right\} \left\{ \left(a_{ir} + \frac{2Y_r d_i \alpha^2 - Y_i Y_r \gamma}{\alpha^3} + \frac{Y_i d_r}{\alpha} + d_i d_r \right) \right. \\ \left. - \left(\frac{Y_i}{\alpha} + d_i \right) \left(\frac{Y_r}{\alpha} + d_r \right) \right\} G^r - \frac{2(\alpha + \gamma)s_{i0}}{(1+s)^2} + \\ \frac{2(b_i - s(\frac{Y_i}{\alpha} + d_i))((1+2s-\rho)r_{00} + 2(\alpha + \gamma)s_0)}{(1+s)^2((1+2s-\rho)(1+s) + 2(b^2 - s^2))} = 0. \end{array} \right. \quad (6.1.2)$$

Corollary 6.1.2. *Let $F = \sqrt[m]{A}$, with $m > 3$, be an m -th root Finsler metric defined on an open subset $U \subset \mathbb{R}^n$, where A is a totally symmetric irreducible function. Consider the Matsumoto type (F, β) -metric given by:*

$$\tilde{F} = F \left(\frac{1}{1 + \frac{\beta}{F}} \right),$$

with $\beta = b_i(x)y^i$. Then $\tilde{F}$ is locally projectively flat if and only if the following condition holds:

$$\left\{ \begin{array}{l} \frac{-m(m-1)}{(1+s)^5} [((1+2s-\rho)(1+s) + 2(b^2 - s^2))(1+2s)] \\ y_i A_0 A^{1-\frac{4}{m}} + \frac{m}{(1+s)^5} [((1+2s-\rho)(1+s) + 2(b^2 - s^2))(1+2s)] \\ (A_{0i} - A_{xi})A^{1-\frac{2}{m}} - \frac{2}{(1+s)^5} \left[((1+2s-\rho)(1+s) + 2(b^2 - s^2))s_{i0} \right. \\ \left. - 2s_0 \left(b_i - s(\frac{Y_i}{\alpha} + d_i) \right) \right] A^{\frac{1}{m}} + \frac{2(1+2s)(b_i - s(\frac{Y_i}{\alpha} + d_i))r_{00}}{(1+s)^5} = 0. \end{array} \right.$$

Theorem 6.1.3. *Let (M, F) be a Douglas space. A Matsumoto type (F, β) -metric*

$$\tilde{F} = (\alpha + \gamma) \phi \left(\frac{\beta}{F} \right),$$

where b_i is the h -vector, is Douglas if and only if the following conditions hold:

$$\left\{ \begin{aligned} H^{ij} &= \left(\frac{-(\alpha + \gamma)}{1 + 2s - \rho} \right) (s_0^i y^j - s_0^j y^i) \\ &+ \left[\frac{(1 + 2s - \rho)r_{00} + 2(\alpha + \gamma)s_0}{\lambda(1 + s)^3(1 + 2s - \rho)} \right] \\ &\times \left\{ b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right\} (b^i y^j - b^j y^i). \end{aligned} \right. \quad (6.1.3)$$

6.2 Preliminaries

Consider the Finsler manifold (M, F) . The associated spray is a global vector field G that is naturally induced on the slit tangent bundle $TM_0 = TM \setminus \{0\}$ by the Finsler metric F . The projection of an integral curve of the spray G onto the base manifold M yields a geodesic on the Finsler manifold (M, F) . For a fixed point $x \in M$, let $F_x := F|_{T_x M}$ denote the restriction of the Finsler metric F to the tangent space at x . To capture the non-Euclidean aspect of the geometry at this point, we define the Cartan torsion at $y \in T_x M \setminus \{0\}$ as the trilinear map:

$$C_y : T_{xM} \otimes T_{xM} \otimes T_{xM} \rightarrow \mathbb{R},$$

given by [118]

$$C_y(u, v, w) := \frac{1}{2} \frac{d}{dt} [g_{y+tw}(u, v)]|_{t=0}, u, v, w \in T_x M.$$

The collection $C := \{C_y\}_{y \in TM_0}$ is called the Cartan torsion of the Finsler manifold. A well-known result states that $C = 0$ if and only if the Finsler metric F is Riemannian.

The Cartan connection on a Finsler manifold M is denoted by $CT = (\Gamma_{jk}^i, N_j^i, C_{jk}^i)$, where

the connection coefficients are defined as follows:

$$\Gamma_{jk}^i = \frac{1}{2}g^{il} \left\{ \frac{\delta g_{jl}}{\delta x^k} + \frac{\delta g_{lk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^l} \right\}, \quad \frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_i^m \frac{\partial}{\partial y^m}, \quad N_j^i = \dot{\partial}_j G^i, \quad N_j^i y^j = 2G^i.$$

∂_i and $\dot{\partial}_i$ represent partial derivatives with respect to the base coordinates x^i and direction coordinates y^i , respectively.

The Cartan connection induces horizontal and vertical covariant derivatives of a tensor field X_j^i as follows:

$$X_{j|k}^i := \frac{\delta X_j^i}{\delta x^k} + X_j^r \Gamma_{rk}^i - X_r^i \Gamma_{jk}^r, \quad X_{j;k}^i := \dot{\partial}_k X_j^i + X_j^r C_{rk}^i - X_r^i C_{jk}^r.$$

For the Cartan connection, we have $g_{ij|k} = 0$ and $g_{ij;k} = 0$. According to the Cartan connection, the h -vector b_i is vertically covariant constant (v -covariant constant) and satisfies the relation $FC_{ij}^h b_h = \rho h_{ij}$, where ρ is a non-zero scalar function, $C_{ij}^h = g^{mh} C_{ijm}$, and h_{ij} are components of the angular metric tensor [106]. The following conditions are satisfied if b_i is an h -vector: (i) $b_{i;j} = 0$, (ii) $FC_{ij}^h b_h = \rho h_{ij}$. Let $c^h = g^{ij} C_{ij}^h$ [106]. Thus, we obtain:

$$\rho = \frac{F}{n-1} c^h b_h, \quad (6.2.1)$$

$$\dot{\partial}_j b_i = \frac{\rho}{F} h_{ij}. \quad (6.2.2)$$

6.3 Matsumoto type (F, β) -metrics

In this chapter, we employ the following notations: $l_i := \dot{\partial}_i F$, $l_{ij} := \dot{\partial}_i \dot{\partial}_j F$, $l_{ijk} := \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k F$. Let b_i be an h -vector in the Finsler space (M, F) such that $h_{ij} y^j = 0$. Furthermore, we have $\dot{\partial}_i \beta = b_i$. Throughout this chapter, contracting any indexed quantity with y^j will be denoted by replacing the free index j with 0. For instance, $b_{i|0}$ denotes $b_{i|j} y^j$.

The following relations are obtained by using (6.2.2), the above notation, and the essential

fact that $l_{i|j} = 0$ and $l_{ijk|l} = 0$.

$$\partial_j b_i = b_{i|j} + \rho N_j^r l_{ir} + b_r \Gamma_{ij}^r, \quad (6.3.1)$$

$$\partial_j l_i = N_j^r l_{ir} + l_r \Gamma_{ij}^r, \quad (6.3.2)$$

$$\partial_k l_{ij} = N_k^r l_{ijr} + l_{rj} \Gamma_{ik}^r + l_{ir} \Gamma_{jk}^r. \quad (6.3.3)$$

Given (6.3.1) and the knowledge that $\partial_k F = l_r N_k^r$, for $s = \frac{\beta}{F}$, we have

$$\dot{\partial}_i s = \frac{1}{F} m_i, \quad \partial_i s = \frac{1}{F} (b_{0|i} + m_r N_i^r), \quad (6.3.4)$$

where $m_i := b_i - s l_i$. The following results are obtained from equations (6.3.1) and (6.3.2):

$$\begin{aligned} \dot{\partial}_k m_i &= (\rho - s) l_{ik} - \frac{1}{F} m_k l_i, \\ \partial_k m_i &= b_{i|k} + (\rho - s) l_{ir} N_k^r - \frac{1}{F} m_r N_k^r l_i + m_r \Gamma_{ik}^r - \frac{1}{F} b_{0|k} l_i. \end{aligned} \quad (6.3.5)$$

Using equations (6.3.4) and (6.3.5), and differentiating (6.1.1) with respect to y^i , y^j , and y^k , it can be shown that

$$\bar{l}_i = \frac{1}{1+s} \left(\frac{y_i}{\alpha} + d_i \right) - \frac{s}{(1+s)^2} \left(b_i - \left(\frac{y_i}{\alpha} + d_i \right) s \right), \quad (6.3.6)$$

$$\left\{ \begin{aligned} \bar{l}_{ij} &= \left(\frac{1+2s-\rho}{(1+s)^2} \right) \left(\frac{a_{ij} \alpha^2 - a_{ij}}{\alpha^3} \right) + \left(\frac{2}{(1+s)^3 (\alpha + \gamma)} \right) \\ &\quad \left(b_i - \left(\frac{y_i}{\alpha} + d_i \right) s \right) \left(b_j - \left(\frac{y_j}{\alpha} + d_j \right) s \right), \end{aligned} \right. \quad (6.3.7)$$

$$\left\{ \begin{aligned} \bar{l}_{ijk} = & \left(\frac{1+2s-\rho}{(1+s)^2} \right) \left(\frac{2\alpha^2 Y_k a_{ij} - (\alpha^2 a_{ij} - a_{ij}) 3Y_k}{\alpha^5} \right) + \frac{2(\rho-s)}{(1+s)^3(\alpha+\gamma)} \\ & \left[\left(b_k - s \left(\frac{Y_k}{\alpha} + d_k \right) \right) \left(\frac{a_{ij}\alpha^2 - a_{ij}}{\alpha^3} \right) + \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \left(\frac{a_{ik}\alpha^2 - a_{ik}}{\alpha^3} \right) + \right. \\ & \left. \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(\frac{a_{jk}\alpha^2 - a_{jk}}{\alpha^3} \right) \right] - \frac{6}{(1+s)^4(\alpha+\gamma)^2} \left[\left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \right. \\ & \left. \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \left(b_k - s \left(\frac{Y_k}{\alpha} + d_k \right) \right) \right] - \frac{2}{(1+s)^3(\alpha+\gamma)^2} \left[\left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \right. \\ & \left. \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \left(\frac{Y_k}{\alpha} + d_k \right) + \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(b_k - s \left(\frac{Y_k}{\alpha} + d_k \right) \right) \right. \\ & \left. \left(\frac{Y_j}{\alpha} + d_j \right) + \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \left(b_k - s \left(\frac{Y_k}{\alpha} + d_k \right) \right) \left(\frac{Y_i}{\alpha} + d_i \right) \right]. \end{aligned} \right. \quad (6.3.8)$$

Definition 6.3.1. [106] A Finsler metric $\tilde{F}$ is called (F, β) -metric if it has the following form $\tilde{F} = F\phi(s)$, $s := \frac{\beta}{F}$, where F is a Finsler metric and $\beta = b_i y^i$ is a 1-form on an n -dimensional manifold M , $\phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ and $\|\beta\|_F < b_0$.

In this context, the (F, β) -metric is referred to as a Matsumoto type (F, β) -metric with h -vector when the associated h -vector is represented by the expression $b_i := b_i(x, y)$.

Lemma 6.3.2. For the Finsler metric $F = \alpha + \gamma$ and 1-form $\beta = b_i y^i$ with h -vector b_i on manifold M with $\|\beta\|_F < b_0$. $\tilde{F} = F\phi(s)$ is a Finsler metric if and only if the positive C^∞ function $\phi(s) = \frac{1}{1+s}$ satisfies:

$$\frac{1+2s-\rho}{(1+s)^2} > 0, \quad \frac{(1+2s-\rho)(1+s)+2(b^2-s^2)}{(1+s)^3} > 0, \quad (6.3.9)$$

when $n \geq 3$ or $\frac{(1+2s-\rho)(1+s)+2(b^2-s^2)}{(1+s)^3} > 0$, when $n = 2$, where $s = \frac{\beta}{F}$ and b are arbitrary numbers with $|s| \leq b < b_0$ and ρ is given by (6.2.1).

Proof. Since the cases $n = 2$ and $n \geq 3$ are identical, the claim is established only for $n \geq 3$. The regularity and positive homogeneity of $\tilde{F}$ can be verified directly. A rigorous convexity verification is presented in the next section. By direct computation, the fundamental tensor

is given by $\tilde{g}_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j \tilde{F}^2$.

$$\left\{ \begin{aligned} \tilde{g}_{ij} = & \left(\frac{1+2s-\rho}{(1+s)^3} \right) \left(a_{ij} + \frac{2Y_j d_i \alpha^2 - Y_i Y_j \gamma}{\alpha^3} + \frac{Y_i d_j}{\alpha} + d_i d_j \right) \\ & + \frac{3b_i b_j}{(1+s)^4} + \left(\frac{-1-4s}{(1+s)^4} \right) \left(\left(\frac{Y_i}{\alpha} + d_i \right) b_j + \left(\frac{Y_j}{\alpha} + d_j \right) b_i \right) \\ & + \left(\frac{(s(1+4s) + \rho(1+s))}{(1+s)^4} \right) \left(\left(\frac{Y_i}{\alpha} + d_i \right) \left(\frac{Y_j}{\alpha} + d_j \right) \right), \end{aligned} \right. \quad (6.3.10)$$

Using [Lemma 1.1.1], we derive

$$\det(\tilde{g}_{ij}) = \left(\frac{1}{1+s} \right)^{n+1} \left(\frac{1+2s-\rho}{(1+s)^2} \right)^{n-2} \left(\frac{1+2s-\rho}{(1+s)^2} + \frac{2(b^2-s^2)}{(1+s)^3} \right) \det(g_{ij}). \quad (6.3.11)$$

Assuming that (6.3.9) is satisfied, we have $\det(\tilde{g}_{ij}) > 0$ using (6.3.9) and (6.3.11). By setting $\rho = 0$, it can be easily shown that Lemma (6.3.2) is satisfied for Matsumoto-type (F, β) -metrics. $\square$

Corollary 6.3.3. *Let M be an n -dimensional manifold. For Finsler metric $F = \alpha + \gamma$ and 1-form $\beta = b_i y^i$ with $\|\beta\|_F < b_0$. $\tilde{F} = (\alpha + \gamma) \left(\frac{1}{1+s} \right)$ is a Matsumoto type Finsler metric if and only if the positive C^∞ function satisfies $\frac{1}{1+s} + \frac{s}{(1+s)^2} > 0$, $\frac{1}{1+s} + \frac{s}{(1+s)^2} + \frac{2(b^2-s^2)}{(1+s)^3} > 0$; when $n \geq 3$, or $\frac{1}{1+s} + \frac{s}{(1+s)^2} + \frac{2(b^2-s^2)}{(1+s)^3} > 0$, when $n = 2$, s and b are arbitrary number.*

The expression for $\tilde{g}^{ij}$ can be obtained from.

$$\tilde{g}^{ij} = \frac{1}{\bar{\rho}} \left[g^{ij} - \frac{\bar{\delta}}{1+b^2\bar{\delta}} b^i b^j - \frac{\bar{\mu}}{1+\bar{\mu}\bar{Y}^2} (l^i + \bar{\lambda} b^i) (l^j + \bar{\lambda} b^j) \right]. \quad (6.3.12)$$

Where $(g^{ij}) = (g_{ij})^{-1} = \left(\frac{\alpha}{\alpha+\gamma} \right) a^{ij} - \left(\frac{\alpha}{(\alpha+\gamma)^2} \right) (D^j y^i + D^i y^j) + \left(\frac{d^2 \alpha + \gamma}{(\alpha+\gamma)^3} \right) y^i y^j$, where $D^j = a^{jr} d_r$, $D^i = a^{ir} d_r$, $d^2 = a^{ir} d_r d_i$, $b^2 = b_i b^i = g^{ij} b_i b_j$, and

$$\bar{\delta} = \frac{3(s(1+4s) + \rho(1+s)) - (1+4s)^2}{(1+s)(s(1+4s) + \rho(1+s))(1+2s-\rho)},$$

$$\bar{\epsilon} = \frac{-(1+4s)}{s(1+4s) + \rho(1+s)},$$

$$\bar{\mu} = \frac{s(1+4s) + \rho(1+s)}{(1+s)(1+2s-\rho)},$$

$$\begin{aligned}\bar{\lambda} &= \frac{-(1+4s)(1+s)(1+2s-\rho) - s(3(s(1+4s) + \rho(1+s)) - (1+4s)^2)}{(1+s)(s(1+4s) + \rho(1+s))(1+2s-\rho) + b^2(3(s(1+4s) + \rho(1+s)) - (1+4s)^2)}, \\ \bar{Y} &= 1+s \left\{ \frac{-(1+4s)}{s(1+4s) + \rho(1+s)} \right. \\ &\quad \left. + \frac{-(1+4s)(1+s)(1+2s-\rho) - s(3(s(1+4s) + \rho(1+s)) - (1+4s)^2)}{(1+s)(s(1+4s) + \rho(1+s))(1+2s-\rho) + b^2(3(s(1+4s) + \rho(1+s)) - (1+4s)^2)} \right\} \\ &\quad + b^2 \left\{ \frac{-(1+4s)}{s(1+4s) + \rho(1+s)} \right\} \\ &\quad \times \left\{ \frac{-(1+4s)(1+s)(1+2s-\rho) - s(3(s(1+4s) + \rho(1+s)) - (1+4s)^2)}{(1+s)(s(1+4s) + \rho(1+s))(1+2s-\rho) + b^2(3(s(1+4s) + \rho(1+s)) - (1+4s)^2)} \right\}.\end{aligned}$$

Differentiating (6.3.10) with respect to y^k , the Cartan tensor $\bar{C}_{ijk}$ is obtained as follows:

$$\bar{C}_{ijk} = \eta C_{ijk} + \frac{\eta'}{2F} h_{ijk} + \frac{\eta'_0}{2F} m_i m_j m_k, \quad (6.3.13)$$

where

$$C_{ijk} := \frac{1}{\alpha^6} \left[(2a_{kj}d_i\alpha^2 + d_i y_j y_k) \alpha^3 - 3\alpha y_k (2y_j d_i \alpha^2 - y_i y_j \gamma) + \alpha^5 a_{ik} d_j - \alpha^3 y_i y_k d_j \right],$$

$$\eta = \frac{1+2s-\rho}{(1+s)^3}, \quad \eta' = \frac{2(1+s) - 3(1+2s-\rho)}{(1+s)^4}, \quad \eta'_0 = \frac{-12}{(1+s)^5},$$

$$\begin{aligned}h_{ijk} &= m_i \left[\left(a_{jk} + \frac{2y_k d_j \alpha^2 - y_j y_k \gamma}{\alpha^3} + \frac{y_j d_k}{\alpha} + d_j d_k \right) - \left(\frac{y_j}{\alpha} + d_j \right) \left(\frac{y_k}{\alpha} + d_k \right) \right] \\ &\quad + m_j \left[\left(a_{ik} + \frac{2y_k d_i \alpha^2 - y_i y_k \gamma}{\alpha^3} + \frac{y_i d_k}{\alpha} + d_i d_k \right) - \left(\frac{y_i}{\alpha} + d_i \right) \left(\frac{y_k}{\alpha} + d_k \right) \right] \\ &\quad + m_k \left[\left(a_{ij} + \frac{2y_j d_i \alpha^2 - y_j y_i \gamma}{\alpha^3} + \frac{y_i d_j}{\alpha} + d_j d_i \right) - \left(\frac{y_j}{\alpha} + d_j \right) \left(\frac{y_i}{\alpha} + d_i \right) \right],\end{aligned}$$

$$\text{and } m_i = b_i - s \left(\frac{y_i}{\alpha} + d_i \right), \quad m_j = b_j - s \left(\frac{y_j}{\alpha} + d_j \right), \quad m_k = b_k - s \left(\frac{y_k}{\alpha} + d_k \right).$$

By employing the operators (6.3.12) and (6.3.13), we can derive the following result:

$$\left\{ \begin{aligned} \bar{C}_{jk}^i &= C_{jk}^i + \frac{\eta'}{2\eta F} h_{jk}^i + \frac{\eta'_0}{2\eta F} m^i m_j m_k - \frac{1}{2\eta F} \left\{ (2\rho\eta + \eta'(b^2 - s^2)) h_{jk} + \right. \\ &\quad \left. (2\eta' + \eta'_0(b^2 - s^2)) m_j m_k \right\} \left\{ \left(\frac{\bar{\delta}}{1+\bar{\delta}b^2} + \frac{\bar{\mu}\bar{\lambda}^2}{1+\bar{\mu}\bar{Y}^2} \right) b^i + \frac{\bar{\mu}\bar{\lambda}^2}{1+\bar{\mu}\bar{Y}^2} l^i \right\}. \end{aligned} \right. \quad (6.3.14)$$

Where $C_{jk}^i = g^{il}C_{ijk}$, $h_{jk}^i = g^{il}h_{ijk}$, $m^i = g^{il}m_i$.

We define $r_{i0} = r_{ij}y^j$, $r_j = b^i r_{ij}$, $r_0 = r_j y^j$, $r_{00} = r_{ij}y^i y^j$, $s_{i0} = s_{ij}y^j$, $s_j = b^i s_{ij}$, $s_0 = s_j y^j$, $s_0^i = g^{ij}s_{j0}$. Then $\partial_k s_{ij} = \frac{1}{2}(\rho_j l_{ik} - \rho_i l_{jk})$, $\partial_k s_{i0} = \frac{1}{2}\rho_0 l_{ik} + s_{ik}$, where $\rho_i = \partial_i \rho$ and $\rho_0 = \rho_k y^k$.

6.4 Spray Coefficients of Matsumoto type (F, β) -Metrics

In this section, we compute the spray coefficients for Matsumoto-type (F, β) -metrics. Differentiating (6.3.6) with respect to x^j and applying (6.3.2) and the second equations in (6.3.4) and (6.3.5), we obtain

$$\left\{ \begin{array}{l} \partial_j \bar{l}_i = \frac{1}{1+s} \left[\left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) N_j^r + \left(\frac{Y_r}{\alpha} + d_r \right) \Gamma_{ij}^r \right] + \frac{2}{(1+s)^3(\alpha+\gamma)} \\ \left[b_{0|j} + \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) N_j^r \right] \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) - \frac{1}{(1+s)^2} \\ \left[b_{i|j} + \left(\rho - s \right) \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) N_j^r + \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \Gamma_{ij}^r \right]. \end{array} \right. \quad (6.4.1)$$

Next, we examine $\bar{l}_{i|j} = 0$, which gives $\partial_j \bar{l}_i = \bar{l}_{ir} \bar{N}_j^r + \bar{l}_r \bar{\Gamma}_{ij}^r$. We first clarify the terms that will be used throughout this research work.

$$D_{jk}^i := \bar{\Gamma}_{jk}^i - \Gamma_{jk}^i, \quad D_j^i := D_{jk}^i y^k = \bar{N}_j^i - N_j^i, \quad D^i := D_j^i y^j = 2\bar{G}^i - 2G^i. \quad (6.4.2)$$

Then

$$\left\{ \begin{array}{l} \partial_j \bar{l}_i = \left\{ \left(\frac{1+2s-\rho}{(1+s)^2} \right) \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) + \frac{2}{(1+s)^3(\alpha+\gamma)} \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \right. \\ \left. \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \right\} \left(D_j^r + N_j^r \right) + \left\{ \frac{1}{1+s} \left(\frac{Y_r}{\alpha} + d_r \right) - \frac{1}{(1+s)^2} \right. \\ \left. \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \right\} \left(D_{ij}^r + \Gamma_{ij}^r \right). \end{array} \right.$$

Adding (6.3.6) and (6.3.7) to the above equation, we get

$$\left\{ \begin{aligned} \partial_j \bar{l}_i &= \left\{ \left(\frac{1+2s-\rho}{(1+s)^2} \right) \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) + \frac{2}{(1+s)^3(\alpha+\gamma)} \left(b_i - s\left(\frac{Y_i}{\alpha} + d_i\right) \right) \right. \\ &\quad \left. \left(b_r - s\left(\frac{Y_r}{\alpha} + d_r\right) \right) \right\} D_j^r + \left\{ \frac{1}{1+s} \left(\frac{Y_r}{\alpha} + d_r \right) - \frac{1}{(1+s)^2} \right. \\ &\quad \left. \left(b_r - s\left(\frac{Y_r}{\alpha} + d_r\right) \right) \right\} D_{ij}^r + \left\{ \left(\frac{1+2s-\rho}{(1+s)^2} \right) \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) + \right. \\ &\quad \left. \frac{2}{(1+s)^3(\alpha+\gamma)} \left(b_i - s\left(\frac{Y_i}{\alpha} + d_i\right) \right) \left(b_r - s\left(\frac{Y_r}{\alpha} + d_r\right) \right) \right\} N_j^r + \\ &\quad \left\{ \left(\frac{1}{1+s} \right) \left(\frac{Y_r}{\alpha} + d_r \right) - \frac{1}{(1+s)^2} \left(b_r - s\left(\frac{Y_r}{\alpha} + d_r\right) \right) \right\} \Gamma_{ij}^r. \end{aligned} \right. \quad (6.4.3)$$

It follows from a comparison of (6.4.1) and (6.4.3) that

$$\left\{ \begin{aligned} \frac{-1}{(1+s)^2} b_{i|j} &= \left\{ \left(\frac{1+2s-\rho}{(1+s)^2} \right) \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) + \frac{2}{(1+s)^3(\alpha+\gamma)} \left(b_i - s\left(\frac{Y_i}{\alpha} + d_i\right) \right) \right. \\ &\quad \left. \left(b_r - s\left(\frac{Y_r}{\alpha} + d_r\right) \right) \right\} D_j^r + \left\{ \frac{1}{1+s} \left(\frac{Y_r}{\alpha} + d_r \right) - \frac{1}{(1+s)^2} \left(b_r - s\left(\frac{Y_r}{\alpha} + d_r\right) \right) \right\} D_{ij}^r \\ &\quad - \frac{2}{(1+s)^3(\alpha+\gamma)} \left\{ \left(b_i - s\left(\frac{Y_i}{\alpha} + d_i\right) \right) b_{0|j} \right\}. \end{aligned} \right.$$

Thus, we have

$$\left\{ \begin{aligned} \frac{-2}{(1+s)^2} r_{ij} &= \left\{ \left(\frac{1+2s-\rho}{(1+s)^2} \right) \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) + \frac{2}{(1+s)^3(\alpha+\gamma)} \right. \\ &\quad \left. \left(b_i - s\left(\frac{Y_i}{\alpha} + d_i\right) \right) \left(b_r - s\left(\frac{Y_r}{\alpha} + d_r\right) \right) \right\} D_j^r + \left\{ \left(\frac{1+2s-\rho}{(1+s)^2} \right) \right. \\ &\quad \left. \left(\frac{a_{jr}\alpha^2 - a_{jr}}{\alpha^3} \right) + \frac{2}{(1+s)^3(\alpha+\gamma)} \left(b_j - s\left(\frac{Y_j}{\alpha} + d_j\right) \right) \right. \\ &\quad \left. \left(b_r - s\left(\frac{Y_r}{\alpha} + d_r\right) \right) \right\} D_i^r + 2 \left\{ \frac{1}{(1+s)} \left(\frac{Y_r}{\alpha} + d_r \right) - \frac{1}{(1+s)^2} \right. \\ &\quad \left. \left(b_r - s\left(\frac{Y_r}{\alpha} + d_r\right) \right) \right\} D_{ij}^r - \frac{2}{(1+s)^3(\alpha+\gamma)} \left\{ \left(b_i - s\left(\frac{Y_i}{\alpha} + d_i\right) \right) \right. \\ &\quad \left. b_{0|j} \left(b_j - s\left(\frac{Y_j}{\alpha} + d_j\right) \right) b_{0|i} \right\}. \end{aligned} \right. \quad (6.4.4)$$

$$\left\{ \begin{aligned} & \frac{-2}{(1+s)^2} s_{ij} = \left\{ \left(\frac{1+2s-\rho}{(1+s)^2} \right) \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) + \frac{2}{(1+s)^3(\alpha+\gamma)} \right. \\ & \left. \left(b_i - s\left(\frac{Y_i}{\alpha} + d_i\right) \right) \left(b_r - s\left(\frac{Y_r}{\alpha} + d_r\right) \right) \right\} D_j^r - \left\{ \left(\frac{1+2s-\rho}{(1+s)^2} \right) \right. \\ & \left. \left(\frac{a_{jr}\alpha^2 - a_{jr}}{\alpha^3} \right) + \frac{2}{(1+s)^3(\alpha+\gamma)} \left\{ \left(b_j - s\left(\frac{Y_j}{\alpha} + d_j\right) \right) \right. \right. \\ & \left. \left. \left(b_r - s\left(\frac{Y_r}{\alpha} + d_r\right) \right) \right\} \right\} D_i^r - \frac{2}{(1+s)^3(\alpha+\gamma)} \left\{ \left(b_i - s\left(\frac{Y_i}{\alpha} + d_i\right) \right) \right. \\ & \left. \left. b_{0|j} - \left(b_j - s\left(\frac{Y_j}{\alpha} + d_j\right) \right) b_{0|i} \right\} \right\}. \end{aligned} \right. \quad (6.4.5)$$

Contracting (6.4.4) and (6.4.5) with y^j , we obtain

$$\left\{ \begin{aligned} & \frac{-2}{(1+s)^2} r_{i0} = \left\{ \left(\frac{1+2s-\rho}{(1+s)^2} \right) \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) + \frac{2}{(1+s)^3(\alpha+\gamma)} \right. \\ & \left. \left(b_i - s\left(\frac{Y_i}{\alpha} + d_i\right) \right) \left(b_r - s\left(\frac{Y_r}{\alpha} + d_r\right) \right) \right\} D^r + 2 \left\{ \frac{1}{1+s} \right. \\ & \left. \left(\frac{Y_r}{\alpha} + d_r \right) - \frac{1}{(1+s)^2} \left(b_r - s\left(\frac{Y_r}{\alpha} + d_r\right) \right) \right\} D_i^r - \frac{2r_{00}}{(1+s)^3(\alpha+\gamma)} \\ & \left. \left(b_i - s\left(\frac{Y_i}{\alpha} + d_i\right) \right) \right\}. \end{aligned} \right. \quad (6.4.6)$$

$$\left\{ \begin{aligned} & \frac{-2}{(1+s)^2} s_{i0} = \left\{ \left(\frac{1+2s-\rho}{(1+s)^2} \right) \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) + \frac{2}{(1+s)^3(\alpha+\gamma)} \right. \\ & \left. \left(b_i - s\left(\frac{Y_i}{\alpha} + d_i\right) \right) \left(b_r - s\left(\frac{Y_r}{\alpha} + d_r\right) \right) \right\} D^r - \\ & \left. \frac{2r_{00}}{(1+s)^3(\alpha+\gamma)} \left(b_i - s\left(\frac{Y_i}{\alpha} + d_i\right) \right) \right\}. \end{aligned} \right. \quad (6.4.7)$$

Upon taking the difference of (6.4.7) and (6.4.6), it follows that

$$\frac{-1}{(1+s)^2} (r_{i0} - s_{i0}) = D_i^r \left\{ \frac{1}{1+s} \left(\frac{Y_r}{\alpha} + d_r \right) - \frac{1}{(1+s)^2} \left(b_r - s\left(\frac{Y_r}{\alpha} + d_r\right) \right) \right\}. \quad (6.4.8)$$

The contraction of (6.4.8) with y^i yields

$$\frac{-1}{(1+s)^2} r_{00} = D^r \left\{ \frac{1}{1+s} \left(\frac{Y_r}{\alpha} + d_r \right) - \frac{1}{(1+s)^2} \left(b_r - s\left(\frac{Y_r}{\alpha} + d_r\right) \right) \right\}. \quad (6.4.9)$$

To determine the spray coefficients of $\tilde{F}$, we first prove the following lemma.

Lemma 6.4.1. *The system of algebraic equations*

$$(i) \left\{ \left(\frac{1+2s-\rho}{(1+s)^2} \right) \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) + \frac{2}{(1+s)^3(\alpha+\gamma)} \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \right. \\ \left. \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \right\} A^r = B_i$$

$$(ii) \left\{ \frac{1}{1+s} \left(\frac{Y_r}{\alpha} + d_r \right) - \frac{1}{(1+s)^2} \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \right\} A^r = B,$$

has a unique solution A^r for given B and B_i satisfying $B_i y^i = 0$, which is given by $A^i = \frac{(\alpha+\gamma)(1+s)^2}{1+2s-\rho} B^i + (1+s) \left(B + \frac{(\alpha+\gamma)B_r b^r}{\lambda(1+s)^2} \right) l^i + \frac{-2(\alpha+\gamma)B_r b^r m^i}{(1+s)(1+2s-\rho)\lambda}$, where $B^i = g^{il} B_l$, $l^i = g^{il} l_i$ and $m^i = g^{il} m_l$.

Proof. Upon contracting (6.3.7) with b^i , it follows that

$$\left\{ \begin{aligned} & b^i \left\{ \left(\frac{1+2s-\rho}{(1+s)^2} \right) \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) + \frac{2}{(1+s)^3(\alpha+\gamma)} \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \right. \\ & \left. \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \right\} = \frac{\lambda}{\alpha+\gamma} \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right), \end{aligned} \right. \quad (6.4.10)$$

where $\lambda := \frac{1+2s-\rho}{(1+s)^2} + \frac{2(b^2-s^2)}{(1+s)^3}$. □

The following is obtained by using (6.4.10) and contracting equation (i) by b^i .

$$\frac{\lambda}{\alpha+\gamma} \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) A^r = B_r b^r \quad (6.4.11)$$

In equation (ii), substituting (6.3.6) results in

$$\frac{1}{1+s} \left(\frac{Y_r}{\alpha} + d_r \right) A^r - \frac{A^r}{(1+s)^2} \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) = B.$$

When we substitute (6.4.11) into this formula, we obtain

$$\left(\frac{Y_r}{\alpha} + d_r \right) A^r = s + 1 \left(B + \frac{\alpha+\gamma}{\lambda(1+s)^2} B_r b^r \right). \quad (6.4.12)$$

Using the formula $l_{ir} = \frac{1}{F} (g_{ir} - l_i l_r)$ and substituting (6.3.7) into equation (i), we obtain

$$\begin{cases} g_{ir} A^r = \frac{(\alpha+\gamma)(1+s)^2}{1+2s-\rho} B_i + \left(\frac{Y_r}{\alpha} + d_r\right) \left(\frac{Y_i}{\alpha} + d_i\right) A^r - \frac{2}{(1+s)(1+2s-\rho)} \\ \left((b_r - s(\frac{Y_r}{\alpha} + d_r)) A^r (b_i - s(\frac{Y_i}{\alpha} + d_i)) \right). \end{cases}$$

The proof is completed by contracting this equation by g^{ij} and applying (6.4.11) and (6.4.12).

Now, we can determine the spray coefficients of $\tilde{F}$.

Using the aforementioned relations and contracting (6.4.7) by b^i , we get $\frac{\lambda}{\alpha+\gamma} (b_r - s(\frac{Y_r}{\alpha} + d_r)) D^r = \frac{-2s_0}{(1+s)^2} + \frac{2r_{00}(b^2-s^2)}{(1+s)^3(\alpha+\gamma)}$. The equations (6.4.7) and (6.4.9) form a system of algebraic equations in $(\frac{Y_r}{\alpha} + d_r) D^r$ and $(b_r - s(\frac{Y_r}{\alpha} + d_r)) D^r$ the solution of which, obtained using Lemma (6.4.1), is given by

$$D^i = \frac{(\alpha+\gamma)(1+s)^2}{1+2s-\rho} B^i + (1+s) \left(B + \frac{(\alpha+\gamma)B_r b^r}{\lambda(1+s)^2} \right) l^i + \frac{-2(\alpha+\gamma)B_r b^r m^i}{(1+s)(1+2s-\rho)\lambda},$$

where $B^i = \frac{-2s_0^i}{(1+s)^2} + \frac{2r_{00}m^i}{(1+s)^3(\alpha+\gamma)}$, $B = \frac{-r_{00}}{(1+s)^2}$, $B_r b^r = \frac{-2s_0}{(1+s)^2} + \frac{2r_{00}(b^2-s^2)}{(1+s)^3(\alpha+\gamma)}$. Since $D^i = 2\tilde{G}^i - 2G^i$, we get the following theorem.

Theorem 6.4.2. Let $\tilde{F}$ be a Matsumoto-type (F, β) -metric with h -vector b_i . Then the spray coefficients of $\tilde{F}$ are given by

$$\begin{cases} 2\tilde{G}^i = 2G^i - \frac{2(\alpha+\gamma)}{1+2s-\rho} s_0^i + \frac{2b^i\{(1+2s-\rho)r_{00}+2(\alpha+\gamma)s_0\}}{(1+s)^3(1+2s-\rho)\lambda} \\ + \left\{ \frac{-(1+4s-\rho)((1+2s-\rho)r_{00}+2(\alpha+\gamma)s_0)}{(1+s)^3(1+2s-\rho)\lambda} \right\} l^i \end{cases} \quad (6.4.13)$$

Corollary 6.4.3. Let $\tilde{F}$ be a Matsumoto type (F, β) -metric. Then the spray coefficients of $\tilde{F}$ are given by

$$\begin{cases} 2\tilde{G}^i = 2G^i - \frac{2(\alpha+\gamma)}{1+2s} s_0^i + \left\{ \frac{(-1-4s)((1+2s)r_{00}+2(\alpha+\gamma)s_0)}{(1+3s+2b^2)(1+2s)} \right\} l^i \\ + \frac{2b^i\{(1+2s)r_{00}+2(\alpha+\gamma)s_0\}}{(1+3s+2b^2)(1+2s)} \end{cases} \quad (6.4.14)$$

6.5 Cartan Connection of Matsumoto type (F, β) -Metrics

In this section, we compute the Cartan connection coefficients of (F, β) -metrics. Using (6.3.4) and (6.3.5), we differentiate (6.3.7) with respect to x^k and obtain

$$\left\{ \begin{aligned} \partial_k \bar{l}_{ij} &= \left(\frac{1+2s-\rho}{(1+s)^2} \right) \partial_k \left(\frac{a_{ij}\alpha^2 - a_{ij}}{\alpha^3} \right) + \frac{2(\rho-s)}{(1+s)^3(\alpha+\gamma)} \left[b_{0|k} + \right. \\ &\quad \left. (b_r - s(\frac{Y_r}{\alpha} + d_r)) N_k^r \right] \left(\frac{a_{ij}\alpha^2 - a_{ij}}{\alpha^3} \right) - \frac{\rho_k}{(1+s)^2} \left(\frac{a_{ij}\alpha^2 - a_{ij}}{\alpha^3} \right) \\ &\quad - \frac{6}{(1+s)^4(\alpha+\gamma)^2} \left[b_{0|k} + (b_r - s(\frac{Y_r}{\alpha} + d_r)) \right] \left(b_i - s(\frac{Y_i}{\alpha} + d_i) \right) \\ &\quad \left(b_j - s(\frac{Y_j}{\alpha} + d_j) \right) - \frac{2}{(1+s)^3(\alpha+\gamma)^2} \left(b_i - s(\frac{Y_i}{\alpha} + d_i) \right) \left(b_j - s(\frac{Y_j}{\alpha} + d_j) \right) \\ &\quad \partial_k (\alpha + \gamma) + \frac{2}{(1+s)^3(\alpha+\gamma)} \left(b_j - s(\frac{Y_j}{\alpha} + d_j) \right) \left[b_{i|k} + (\rho - s) \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) \right. \\ &\quad \left. N_k^r - \frac{1}{\alpha+\gamma} (b_r - s(\frac{Y_r}{\alpha} + d_r)) N_k^r \left(\frac{Y_i}{\alpha} + d_i \right) + (b_r - s(\frac{Y_r}{\alpha} + d_r)) \Gamma_{ik}^r \right. \\ &\quad \left. - \frac{1}{\alpha+\gamma} b_{0|k} \left(\frac{Y_i}{\alpha} + d_i \right) \right] + \frac{2}{(1+s)^3(\alpha+\gamma)} \left(b_i - s(\frac{Y_i}{\alpha} + d_i) \right) \left[b_{j|k} + (\rho - s) \right. \\ &\quad \left. \left(\frac{a_{rj}\alpha^2 - a_{rj}}{\alpha^3} \right) N_j^r - \frac{1}{\alpha+\gamma} (b_r - s(\frac{Y_r}{\alpha} + d_r)) N_k^r \left(\frac{Y_j}{\alpha} + d_j \right) + \right. \\ &\quad \left. \left. (b_r - s(\frac{Y_r}{\alpha} + d_r)) \Gamma_{jk}^r - \frac{1}{\alpha+\gamma} b_{0|k} \left(\frac{Y_j}{\alpha} + d_j \right) \right] \right]. \end{aligned} \right. \quad (6.5.1)$$

Using the formula $\bar{l}_{ij|k} = 0$, that is $\partial_k \bar{l}_{ij} = \bar{l}_{ijr} \bar{N}_k^r + \bar{l}_{rj} \bar{\Gamma}_{ik}^r + \bar{l}_{ir} \bar{\Gamma}_{jk}^r$, and applying (6.4.2), we have $\partial_k \bar{l}_{ij} = \bar{l}_{ijr} (D_k^r + N_k^r) + \bar{l}_{rj} (D_{ik}^r + \Gamma_{ik}^r) + \bar{l}_{ir} (D_{jk}^r + \Gamma_{jk}^r)$. Substituting the values of $\bar{l}_{ir}$, $\bar{l}_{rj}$ and $\bar{l}_{ijr}$ from (6.3.7) and (6.3.8) into the above equation yields

$$\left\{ \begin{aligned} \partial_k \bar{l}_{ij} &= \left\{ \left(\frac{1+2s-\rho}{(1+s)^2} \right) \left(\frac{2\alpha^2 Y_r a_{ij} - (\alpha^2 a_{ij} - a_{ij}) 3Y_r}{\alpha^5} \right) + \frac{2(\rho-s)}{(1+s)^3(\alpha+\gamma)} \right. \\ &\quad \left((b_r - s(\frac{Y_r}{\alpha} + d_r)) \left(\frac{a_{ij}\alpha^2 - a_{ij}}{\alpha^3} \right) \left(b_j - s(\frac{Y_j}{\alpha} + d_j) \right) \right. \\ &\quad \left. \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) + \left(b_i - s(\frac{Y_i}{\alpha} + d_i) \right) \left(\frac{a_{jr}\alpha^2 - a_{jr}}{\alpha^3} \right) \right) - \frac{6}{(1+s)^4(\alpha+\gamma)^2} \\ &\quad \left. \left\{ \left(b_i - s(\frac{Y_i}{\alpha} + d_i) \right) \left(b_j - s(\frac{Y_j}{\alpha} + d_j) \right) \right\} \right\} \end{aligned} \right. \quad (6.5.2)$$

$$\left\{ \begin{aligned}
& \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \left\{ - \frac{2}{(1+s)^3(\alpha+\gamma)^2} \left[\left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \right. \right. \\
& \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \left(\frac{Y_r}{\alpha} + d_r \right) + \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \\
& \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \left(\frac{Y_j}{\alpha} + d_j \right) + \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \\
& \left. \left. \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \left(\frac{Y_i}{\alpha} + d_i \right) \right] \right\} D_k^r + \left\{ \frac{1+2s-\rho}{(1+s)^2} \left(\frac{a_{rj}\alpha^2 - a_{rj}}{\alpha^3} \right) \right. \\
& + \frac{2(b_r - s(\frac{Y_r}{\alpha} + d_r))(b_j - s(\frac{Y_j}{\alpha} + d_j))}{(1+s)^3(\alpha+\gamma)} \left\{ D_{ik}^r + \left\{ \frac{1+2s-\rho}{(1+s)^2} \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) + \right. \right. \\
& \left. \left. \frac{2(b_i - s(\frac{Y_i}{\alpha} + d_i))(b_r - s(\frac{Y_r}{\alpha} + d_r))}{(1+s)^3(\alpha+\gamma)} \right\} D_{jk}^r + \left[\left\{ \left(\frac{1+2s-\rho}{(1+s)^2} \right) \right. \right. \right. \\
& \left. \left. \left(\frac{2\alpha^2 Y_r a_{ij} - (\alpha^2 a_{ij} - a_{ij}) 3Y_r}{\alpha^5} \right) \right\} + \frac{2(\rho-s)}{(1+s)^3(\alpha+\gamma)} \left[\left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \right. \right. \\
& \left. \left(\frac{a_{ij}\alpha^2 - a_{ij}}{\alpha^3} \right) + \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) + \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \right. \\
& \left. \left. \left(\frac{a_{jr}\alpha^2 - a_{jr}}{\alpha^3} \right) \right] - \frac{6}{(1+s)^4(\alpha+\gamma)^2} \left\{ \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \right. \right. \\
& \left. \left. \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \right\} - \frac{2}{(1+s)^3(\alpha+\gamma)^2} \left[\left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \right. \right. \\
& \left. \left(\frac{Y_r}{\alpha} + d_r \right) + \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \left(\frac{Y_j}{\alpha} + d_j \right) + \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \right. \\
& \left. \left. \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \left(\frac{Y_i}{\alpha} + d_i \right) \right] \right] N_k^r + \Gamma_{ik}^r \left\{ \left(\frac{1+2s-\rho}{(1+s)^2} \right) \left(\frac{a_{rj}\alpha^2 - a_{rj}}{\alpha^3} \right) \right. \right. \\
& + \frac{2}{(1+s)^3} \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \left. \right\} + \Gamma_{jk}^r \left\{ \left(\frac{1+2s-\rho}{(1+s)^2} \right) \right. \\
& \left. \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) + \frac{2}{(1+s)^3(\alpha+\gamma)} \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \right\}.
\end{aligned} \right.$$

The following is obtained by comparing (6.5.1) and (6.5.2) and applying (6.3.3) and the fact that $\partial_k F = l_r N_k^r$.

$$\left\{ \begin{aligned}
& \left[\frac{(1+2s-\rho)}{(1+s)^2} \left(\frac{2\alpha^2 Y_r a_{ij} - 3Y_r(\alpha^2 a_{ij} - a_{ij})}{\alpha^5} \right) + \frac{2(\rho-s)}{(1+s)^3(\alpha+\gamma)} \right. \\
& \left[\left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \left(\frac{a_{ij}\alpha^2 - a_{ij}}{\alpha^3} \right) + \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \right. \\
& \left. \left. \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) + \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(\frac{a_{jr}\alpha^2 - a_{jr}}{\alpha^3} \right) \right] - \frac{6}{(1+s)^4(\alpha+\gamma)^2}
\end{aligned} \right. \quad (6.5.3)$$

$$\left\{ \begin{aligned} & \left[\left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \right] \\ & - \frac{2}{(1+s)^3(\alpha+\gamma)^2} \left[\left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \left(\frac{Y_r}{\alpha} + d_r \right) \right. \\ & + \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \left(\frac{Y_j}{\alpha} + d_j \right) + \left(\frac{Y_i}{\alpha} + d_i \right) \\ & \left. \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \right] D_k^r + \left[\frac{(1+2s-\rho)a_{rj}(\alpha^2-1)}{(1+s)^2\alpha^3} \right. \\ & + \frac{2(b_r-s(\frac{Y_r}{\alpha}+d_r))(b_j-s(\frac{Y_j}{\alpha}+d_j))}{(1+s)^3(\alpha+\gamma)} \left. \right] D_{ik}^r + \left[\frac{(1+2s-\rho)a_{ir}(\alpha^2-1)}{(1+s)^2\alpha^3} \right. \\ & + \frac{2(b_r-s(\frac{Y_r}{\alpha}+d_r))(b_i-s(\frac{Y_i}{\alpha}+d_i))}{(1+s)^3(\alpha+\gamma)} \left. \right] D_{jk}^r = \frac{-\rho_k(a_{ij}\alpha^2-a_{ij})}{\alpha^3(1+s)^2} + \\ & \frac{2(\rho-s)b_{0|k}}{(1+s)^3(\alpha+\gamma)} \left(\frac{\alpha^2 a_{ij}-a_{ij}}{\alpha^3} \right) + \frac{2}{(1+s)^3(\alpha+\gamma)} \left[\left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) b_{i|k} \right. \\ & + \left. \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) b_{j|k} \right] - \frac{2b_{0|k}}{(1+s)^3(\alpha+\gamma)^2} \left[\left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(\frac{Y_j}{\alpha} + d_j \right) \right. \\ & + \left. \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \left(\frac{Y_i}{\alpha} + d_i \right) \right] - \frac{6}{(1+s)^4(\alpha+\gamma)^2} b_{0|k} \left[\left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \right. \\ & \left. \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \right]. \end{aligned} \right.$$

Contracting (6.5.3) by y^k yields

$$\left\{ \begin{aligned} & \left[\frac{(1+2s-\rho)}{(1+s)^2} \left(\frac{2\alpha^2 Y_r a_{ij} - 3Y_r(\alpha^2 a_{ij} - a_{ij})}{\alpha^5} \right) + \frac{2(\rho-s)}{(1+s)^3(\alpha+\gamma)} \right. \\ & \left[\left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \left(\frac{a_{ij}\alpha^2 - a_{ij}}{\alpha^3} \right) + \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \right. \\ & \left. \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) + \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(\frac{a_{jr}\alpha^2 - a_{jr}}{\alpha^3} \right) \right] - \frac{6}{(1+s)^4(\alpha+\gamma)^2} \\ & \left[\left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \right] \\ & - \frac{2}{(1+s)^3(\alpha+\gamma)^2} \left[\left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \left(\frac{Y_r}{\alpha} + d_r \right) \right. \\ & + \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \left(\frac{Y_j}{\alpha} + d_j \right) + \left(\frac{Y_i}{\alpha} + d_i \right) \\ & \left. \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \right] D^r + \left[\frac{(1+2s-\rho)a_{rj}(\alpha^2-1)}{(1+s)^2\alpha^3} \right. \\ & + \frac{2(b_r-s(\frac{Y_r}{\alpha}+d_r))(b_j-s(\frac{Y_j}{\alpha}+d_j))}{(1+s)^3(\alpha+\gamma)} \left. \right] D_i^r + \left[\frac{(1+2s-\rho)a_{ir}(\alpha^2-1)}{(1+s)^2\alpha^3} \right. \\ & + \frac{2(b_r-s(\frac{Y_r}{\alpha}+d_r))(b_i-s(\frac{Y_i}{\alpha}+d_i))}{(1+s)^3(\alpha+\gamma)} \left. \right] D_j^r = \frac{-\rho_0(a_{ij}\alpha^2-a_{ij})}{\alpha^3(1+s)^2} + \\ & \frac{2(\rho-s)r_{00}}{(1+s)^3(\alpha+\gamma)} \left(\frac{\alpha^2 a_{ij}-a_{ij}}{\alpha^3} \right) + \frac{2}{(1+s)^3(\alpha+\gamma)} \end{aligned} \right. \quad (6.5.4)$$

$$\left\{ \begin{aligned} & \left[\left(b_j - s\left(\frac{Y_j}{\alpha} + d_j\right) \right) b_{i|0} + \left(b_i - s\left(\frac{Y_i}{\alpha} + d_i\right) \right) b_{j|0} \right] - \frac{2r_{00}}{(1+s)^3(\alpha+\gamma)^2} \\ & \left[\left(b_i - s\left(\frac{Y_i}{\alpha} + d_i\right) \right) \left(\frac{Y_j}{\alpha} + d_j \right) + \left(b_j - s\left(\frac{Y_j}{\alpha} + d_j\right) \right) \left(\frac{Y_i}{\alpha} + d_i \right) \right] \\ & - \frac{6}{(1+s)^4(\alpha+\gamma)^2} r_{00} \left[\left(b_i - s\left(\frac{Y_i}{\alpha} + d_i\right) \right) \left(b_j - s\left(\frac{Y_j}{\alpha} + d_j\right) \right) \right]. \end{aligned} \right.$$

Equation (6.5.4) implies that upon substituting (6.4.5),

$$\left\{ \begin{aligned} & \left\{ \left(\frac{1+2s-\rho}{(1+s)^2} \right) \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) + \frac{2}{(1+s)^3(\alpha+\gamma)} \left(b_i - s\left(\frac{Y_i}{\alpha} + d_i\right) \right) \right. \\ & \left. \left(b_r - s\left(\frac{Y_r}{\alpha} + d_r\right) \right) \right\} D_j^r = Q_{ij}, \end{aligned} \right. \quad (6.5.5)$$

where,

$$\left\{ \begin{aligned} Q_{ij} := & \frac{-1}{2} \left[\frac{(1+2s-\rho)}{(1+s)^2} \left(\frac{2\alpha^2 Y_r a_{ij} - 3Y_r(\alpha^2 a_{ij} - a_{ij})}{\alpha^5} \right) + \frac{2(\rho-s)}{(1+s)^3(\alpha+\gamma)} \right. \\ & \left[\left(b_r - s\left(\frac{Y_r}{\alpha} + d_r\right) \right) \left(\frac{a_{ij}\alpha^2 - a_{ij}}{\alpha^3} \right) + \left(b_j - s\left(\frac{Y_j}{\alpha} + d_j\right) \right) \right. \\ & \left. \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) + \left(b_i - s\left(\frac{Y_i}{\alpha} + d_i\right) \right) \left(\frac{a_{jr}\alpha^2 - a_{jr}}{\alpha^3} \right) \right] - \frac{6}{(1+s)^4(\alpha+\gamma)^2} \\ & \left[\left(b_i - s\left(\frac{Y_i}{\alpha} + d_i\right) \right) \left(b_j - s\left(\frac{Y_j}{\alpha} + d_j\right) \right) \left(b_r - s\left(\frac{Y_r}{\alpha} + d_r\right) \right) \right] \\ & - \frac{2}{(1+s)^3(\alpha+\gamma)^2} \left[\left(b_i - s\left(\frac{Y_i}{\alpha} + d_i\right) \right) \left(b_j - s\left(\frac{Y_j}{\alpha} + d_j\right) \right) \left(\frac{Y_r}{\alpha} + d_r \right) \right. \\ & + \left(b_i - s\left(\frac{Y_i}{\alpha} + d_i\right) \right) \left(b_r - s\left(\frac{Y_r}{\alpha} + d_r\right) \right) \left(\frac{Y_j}{\alpha} + d_j \right) + \left(\frac{Y_i}{\alpha} + d_i \right) \\ & \left. \left(b_j - s\left(\frac{Y_j}{\alpha} + d_j\right) \right) \left(b_r - s\left(\frac{Y_r}{\alpha} + d_r\right) \right) \right] \Big] D^r. \end{aligned} \right.$$

$$\left\{ \begin{aligned} & -\frac{1}{(1+s)^2} s_{ij} - \frac{\rho_0}{2(1+s)^2} \left(\frac{\alpha^2 a_{ij} - a_{ij}}{\alpha^3} \right) \\ & + \frac{2}{(1+s)^3(\alpha+\gamma)} \left[\left(b_i - s\left(\frac{Y_i}{\alpha} + d_i\right) \right) r_{j0} \right. \\ & \left. + \left(b_j - s\left(\frac{Y_j}{\alpha} + d_j\right) \right) s_{i0} \right] + \frac{(\rho-s)r_{00}}{(1+s)^3(\alpha+\gamma)} \\ & \left(\frac{a_{ij}\alpha^2 - a_{ij}}{\alpha^3} \right) \end{aligned} \right.$$

$$\left\{ \begin{array}{l} -\frac{r_{00}}{(1+s)^3(\alpha+\gamma)^2} \left[\left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \right. \\ \left. \left(\frac{Y_j}{\alpha} + d_j \right) + \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \left(\frac{Y_i}{\alpha} + d_i \right) \right] \\ -\frac{3r_{00}}{(1+s)^4(\alpha+\gamma)^2} \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \end{array} \right\}.$$

From (6.4.7), we have $Q_{ij}y^i = 0$. Furthermore, equation (6.4.8) can be written as

$$\left\{ \frac{1}{1+s} \left(\frac{Y_r}{\alpha} + d_r \right) - \frac{1}{(1+s)^2} \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \right\} D_j^r = Q_j, \quad (6.5.6)$$

where $Q_j = \frac{-1}{(1+s)^2} (r_{j0} - s_{j0})$. The system of algebraic equations, consisting of (6.5.6) and (6.5.5), the solution of which is given by Lemma (6.4.1), is

$$D_j^i = \frac{(\alpha+\gamma)(1+s)^2}{1+2s-\rho} Q_j^i + (s+1) \left(Q_j + \frac{(\alpha+\gamma)Q_{rj}b^r}{\lambda(1+s)^2} \right) l^i - \frac{2(\alpha+\gamma)(Q_{rj}b^r m^i)}{\lambda(1+s)(1+2s-\rho)}.$$

Where $Q_j^i = g^{ir} Q_{rj}$. Then by (6.4.2) we have

$$\left\{ \begin{array}{l} \bar{N}_j^i = N_j^i + \frac{(\alpha+\gamma)(1+s)^2}{1+2s-\rho} Q_j^i + (s+1) l^i \left(Q_j + \frac{(\alpha+\gamma)Q_{rj}b^r}{\lambda(1+s)^2} \right) \\ - \frac{2(\alpha+\gamma)(Q_{rj}b^r m^i)}{\lambda(1+s)(1+2s-\rho)}. \end{array} \right. \quad (6.5.7)$$

Finally, cyclically permuting the indices i, j , and k in equation (6.5.3) yields

$$\bar{l}_{rj} D_{ik}^r = M_{jik}, \quad (6.5.8)$$

where

$$\left\{ \begin{array}{l} M_{jik} = -\frac{1}{2} \left[\left\{ \left(\frac{1+2s-\rho}{(1+s)^2} \right) \left(\frac{2\alpha^2 Y_r a_{ij} - 3Y_r (\alpha^2 a_{ij} - a_{ij})}{\alpha^5} \right) + \frac{2(\rho-s)}{(1+s)^3(\alpha+\gamma)} \right. \right. \\ \left. \left\{ \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \left(\frac{a_{ij}\alpha^2 - a_{ij}}{\alpha^3} \right) + \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \right. \right. \\ \left. \left. \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) + \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(\frac{a_{jr}\alpha^2 - a_{jr}}{\alpha^3} \right) \right\} - \frac{6}{(1+s)^4(\alpha+\gamma)^2} \right] \end{array} \right.$$

$$\left\{ \begin{aligned}
& \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \Big\} \\
& - \frac{2}{(1+s)^3(\alpha+\gamma)^2} \left[\left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \right. \\
& \left. \left(\frac{Y_r}{\alpha} + d_r \right) + \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \left(\frac{Y_j}{\alpha} + d_j \right) \right. \\
& \left. + \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \left(\frac{Y_i}{\alpha} + d_i \right) \right] \Big\} D_k^r + \left[\left(\frac{1+2s-\rho}{(1+s)^2} \right) \right. \\
& \left(\frac{2\alpha^2 Y_r a_{jk} - 3Y_r(\alpha^2 a_{jk} - a_{jk})}{\alpha^5} \right) + \frac{2(\rho-s)}{(1+s)^3(\alpha+\gamma)} \left\{ \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \right. \\
& \left(\frac{a_{jk}\alpha^2 - a_{jk}}{\alpha^3} \right) + \left(b_k - s \left(\frac{Y_k}{\alpha} + d_k \right) \right) \left(\frac{a_{jr}\alpha^2 - a_{jr}}{\alpha^3} \right) + \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \\
& \left(\frac{a_{kr}\alpha^2 - a_{kr}}{\alpha^3} \right) \Big\} - \frac{6}{(1+s)^4(\alpha+\gamma)^2} \left\{ \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \left(b_k - s \left(\frac{Y_k}{\alpha} + d_k \right) \right) \right. \\
& \left. \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \right\} - \frac{2}{(1+s)^3(\alpha+\gamma)^2} \left\{ \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \right. \\
& \left(b_k - s \left(\frac{Y_k}{\alpha} + d_k \right) \right) \left(\frac{Y_r}{\alpha} + d_r \right) + \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \\
& \left(\frac{Y_k}{\alpha} + d_k \right) + \left(b_k - s \left(\frac{Y_k}{\alpha} + d_k \right) \right) \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \left(\frac{Y_j}{\alpha} + d_j \right) \Big\} \Big] D_i^r \\
& - \left[\left(\frac{1+2s-\rho}{(1+s)^2} \right) \left(\frac{2\alpha^2 Y_r a_{ki} - 3Y_r(\alpha^2 a_{ki} - a_{ki})}{\alpha^5} \right) + \frac{2(\rho-s)}{(1+s)^3(\alpha+\gamma)} \right. \\
& \left\{ \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \left(\frac{a_{ki}\alpha^2 - a_{ki}}{\alpha^3} \right) + \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(\frac{a_{kr}\alpha^2 - a_{kr}}{\alpha^3} \right) \right. \\
& + \left(b_k - s \left(\frac{Y_k}{\alpha} + d_k \right) \right) \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) \Big\} - \frac{6}{(1+s)^4(\alpha+\gamma)^2} \left\{ \left(b_k - s \left(\frac{Y_k}{\alpha} + d_k \right) \right) \right. \\
& \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \Big\} - \frac{2}{(1+s)^3(\alpha+\gamma)^2} \left\{ \left(b_k - s \left(\frac{Y_k}{\alpha} + d_k \right) \right) \right. \\
& \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(\frac{Y_r}{\alpha} + d_r \right) + \left(b_k - s \left(\frac{Y_k}{\alpha} + d_k \right) \right) \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \\
& \left(\frac{Y_i}{\alpha} + d_i \right) + \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \left(\frac{Y_k}{\alpha} + d_k \right) \Big\} \Big] D_j^r \\
& - \frac{1}{2(1+s)^2} \left[\frac{\rho_k(a_{ij}\alpha^2 - a_{ij})}{\alpha^3} + \frac{\rho_i(a_{jk}\alpha^2 - a_{jk})}{\alpha^3} - \frac{\rho_j(a_{ik}\alpha^2 - a_{ik})}{\alpha^3} \right] + \left(\frac{2}{(1+s)^3(\alpha+\gamma)} \right) \\
& \left[\left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) r_{ik} + \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) s_{jk} + \left(b_k - s \left(\frac{Y_k}{\alpha} + d_k \right) \right) s_{ji} \right] \\
& + \frac{\rho-s}{(1+s)^3(\alpha+\gamma)} \left[b_{0|k} \left(\frac{a_{ij}\alpha^2 - a_{ij}}{\alpha^3} \right) + b_{0|i} \left(\frac{a_{jk}\alpha^2 - a_{jk}}{\alpha^3} \right) - b_{0|j} \left(\frac{a_{ik}\alpha^2 - a_{ik}}{\alpha^3} \right) \right] \\
& - \frac{3}{(1+s)^4(\alpha+\gamma)^2} \left[b_{0|k} \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) + b_{0|i} \left(b_k - s \left(\frac{Y_k}{\alpha} + d_k \right) \right) \right. \\
& \left. \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) - b_{0|j} \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(b_k - s \left(\frac{Y_k}{\alpha} + d_k \right) \right) \right] - \\
& \left(\frac{1}{(1+s)^3(\alpha+\gamma)^2} \right) \left[b_{0|k} \left\{ \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(\frac{Y_j}{\alpha} + d_j \right) + \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \right. \right. \\
& \left. \left(\frac{Y_i}{\alpha} + d_i \right) \right\} + b_{0|i} \left\{ \left(b_j - s \left(\frac{Y_j}{\alpha} + d_j \right) \right) \left(\frac{Y_k}{\alpha} + d_k \right) + \left(b_k - s \left(\frac{Y_k}{\alpha} + d_k \right) \right) \left(\frac{Y_j}{\alpha} + d_j \right) \right\} \right. \\
& \left. - b_{0|j} \left\{ \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \left(\frac{Y_k}{\alpha} + d_k \right) + \left(b_k - s \left(\frac{Y_k}{\alpha} + d_k \right) \right) \left(\frac{Y_i}{\alpha} + d_i \right) \right\} \right].
\end{aligned} \right\}$$

Furthermore, by (6.5.4) we obtain $M_{jik}y^j = 0$. Additionally, equation (6.4.4) can be written as

$$\bar{l}_r D_{ik}^r = M_{ik}, \quad (6.5.9)$$

where $M_{ik} = \frac{-r_{ik}}{(1+s)^2} - \frac{1}{2} \left\{ \frac{(1+2s-\rho)a_{ir}(\alpha^2-1)}{(1+s)^2\alpha^3} + \frac{2(b_i-s(\frac{Y_i}{\alpha}+d_i))(b_r-s(\frac{Y_r}{\alpha}+d_r))}{(1+s)^3(\alpha+\gamma)} \right\} D_k^r$
 $\frac{-D_i^r}{2} \left\{ \frac{(1+2s-\rho)a_{rk}(\alpha^2-1)}{(1+s)^2\alpha^3} + \frac{2(b_r-s(\frac{Y_r}{\alpha}+d_r))(b_k-s(\frac{Y_k}{\alpha}+d_k))}{(1+s)^3(\alpha+\gamma)} \right\} + \frac{1}{(1+s)^3(\alpha+\gamma)}$
 $\left\{ (b_i-s(\frac{Y_i}{\alpha}+d_i))b_{0|k} + (b_k-s(\frac{Y_k}{\alpha}+d_k))b_{0|i} \right\}$. Using Lemma (6.4.1) with equations (6.5.8) and (6.5.9) implies that

$$\begin{cases} D_{jk}^i = \frac{(\alpha+\gamma)(1+s)^2}{1+2s-\rho} M_{jk}^i + (s+1)l^i \left(M_{jk} + \frac{(\alpha+\gamma)M_{rjk}b^r}{\lambda(1+s)^2} \right) \\ - \frac{2(\alpha+\gamma)M_{rjk}b^r m^i}{\lambda(1+2s-\rho)(1+s)} \end{cases}$$

Where, $M_{jk}^i = g^{ir}Q_{rjk}$. Then by (6.4.2) we get

$$\begin{cases} \bar{\Gamma}_{jk}^i = \Gamma_{jk}^i + \left(\frac{(\alpha+\gamma)(1+s)^2}{1+2s-\rho} \right) M_{jk}^i + (s+1) \left(M_{jk} + \frac{(\alpha+\gamma)M_{rjk}b^r}{\lambda(1+s)^2} \right) l^i \\ - \frac{2(\alpha+\gamma)M_{rjk}b^r m^i}{\lambda(1+2s-\rho)(1+s)}. \end{cases} \quad (6.5.10)$$

Theorem 6.5.1. Let $C\bar{\Gamma} = (\bar{\Gamma}_{jk}^i, \bar{N}_j^i, \bar{C}_{jk}^i)$ be the Cartan connection for the Finsler space $(M, \tilde{F})$. Thus, the Cartan connection is completely determined by equations (6.3.14), (6.5.7), and (6.5.10).

6.6 Proof of Theorem 6.1.1

Let F and $\tilde{F}$ be projectively related, that is, $\bar{G}^i - G^i = Py^i$, where $\bar{G}^i$ and G^i are the geodesic spray coefficients of $\tilde{F}$ and F , respectively, and $P = P(x, y)$ is a scalar function on the slit tangent bundle TM_0 [106]. Equation (6.4.2) implies that $D^i = 2Py^i$. Substituting this into

(6.4.13) yields

$$\begin{cases} 2Py^i = \frac{-2(\alpha+\gamma)s_0^i}{1+2s-\rho} + \left[\frac{(-1-4s+\rho)\{(1+2s-\rho)r_{00}+2(\alpha+\gamma)s_0\}}{(1+s)^3(1+2s-\rho)\lambda} \right] l^i \\ + \frac{2b^i((1+2s-\rho)r_{00}+2(\alpha+\gamma)s_0)}{(1+s)^3(1+2s-\rho)\lambda}. \end{cases} \quad (6.6.1)$$

Contracting (6.6.1) by $y_i = g_{ij}y^j$ and using the facts that $s_0^i y_i = 0$ and $l^i y_i = F$, we obtain $P = \frac{-[(1+2s-\rho)r_{00}+2(\alpha+\gamma)s_0]}{2(\alpha+\gamma)\lambda(1+s)^3}$. Now, suppose $\tilde{F}$ is projectively flat, then $2\tilde{G}^i = 2G^i + D^i = 2\bar{P}y^i$. Using the same calculations as above, by (6.4.13) we obtain

$$\begin{aligned} G^j \left\{ \left(a_{ij} + \frac{2Y_j d_i \alpha^2 - Y_i Y_j \gamma}{\alpha^3} + \frac{Y_i d_j}{\alpha} + d_i d_j \right) - \left(\frac{Y_i}{\alpha} + d_i \right) \left(\frac{Y_j}{\alpha} + d_j \right) \right\} \\ - \frac{(\alpha+\gamma)s_{i0}}{1+2s-\rho} + \left[\frac{(1+2s-\rho)r_{00}+2(\alpha+\gamma)s_0}{\lambda(1+s)^3(1+2s-\rho)} \right] \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) = 0. \end{aligned}$$

Next, substituting (6.1.2) into (6.4.13) gives

$$\begin{cases} \bar{G}^i = G^i + \left(\frac{-(\alpha+\gamma)s_{r0}}{1+2s-\rho} + \frac{(1+2s-\rho)r_{00}+2(\alpha+\gamma)s_0}{\lambda(1+s)^3(1+2s-\rho)} \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \right) g^{ri} \\ - \frac{l^i}{2\lambda(1+s)^3} [(1+2s-\rho)r_{00}+2(\alpha+\gamma)s_0] \\ = G^i - \left\{ \left(a_{rj} + \frac{2Y_j d_r \alpha^2 - Y_j Y_r \gamma}{\alpha^3} + \frac{Y_j d_r}{\alpha} + d_j d_r \right) - \left(\frac{Y_r}{\alpha} + d_r \right) \left(\frac{Y_j}{\alpha} + d_j \right) \right\} g^{ri} G^j \\ - \frac{l^i}{2\lambda(1+s)^3} [(1+2s-\rho)r_{00}+2(\alpha+\gamma)s_0] \\ = l^i \left(\left(\frac{Y_j}{\alpha} + d_j \right) G^j - \frac{1}{2\lambda(1+s)^3} [(1+2s-\rho)r_{00}+2(\alpha+\gamma)s_0] \right) = \bar{P}y^i. \end{cases}$$

Thus, the proof is established.

6.6.1 Proof of Corollary 6.1.2

We note that for m -root Finsler metrics we have [106]:

$$A_i = \frac{\partial A}{\partial y^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}, \quad A_{x^i} = \frac{\partial A}{\partial x^i}, \quad A_0 = A_{x^i} y^i, \quad A_{0l} = A_{x^r y^l} y^r, \quad (6.6.2)$$

and $2G^i = A^{ir}(A_{0r} - A_{x^r})$. Also, it is not hard to get $A_i = mA^{1-\frac{2}{m}}y_i$, and $A_i^r = (mA^{1-\frac{2}{m}}y^r) = mA^{1-\frac{2}{m}}\left(\delta_i^r + (m-2)\left(\frac{Y_i}{\alpha} + d_i\right)l^r\right)$. Then after some calculations we have [106]

$$\begin{aligned} & 2\left\{\left(a_{ij} + \frac{2Y_j d_i \alpha^2 - Y_i Y_j \gamma}{\alpha^3} + \frac{Y_i d_j}{\alpha} + d_i d_j\right) - \left(\frac{Y_i}{\alpha} + d_i\right)\left(\frac{Y_j}{\alpha} + d_j\right)\right\}G^j \\ &= mA^{1-\frac{2}{m}}\left(A_{0i} - A_{x^i} - (m-1)A_0 A^{\frac{-1}{m}}\left(\frac{Y_i}{\alpha} + d_i\right)\right). \end{aligned} \quad (6.6.3)$$

Combining the preceding equations with (6.1.2) shows that

$$-m(m-1)A_0 y_i A^{1-\frac{4}{m}} + m(A_{0i} - A_{x^i})A^{1-\frac{2}{m}} + 2s_{i0}A^{\frac{1}{m}} = 0.$$

The preceding equation yields $A_{x^i} = 0$ and $s_{ij} = 0$ for $m \neq 5$. For $m = 5$, the same result can be obtained by separating the rational and irrational parts of the equation.

Lemma 6.6.1. *Let $F = \sqrt[m]{A}$ ($m > 2, m \neq 5$) be an m -th root Finsler metric on an open subset $U \subset \mathbb{R}^n$ [106]. Suppose that the equation $\psi A^{1-\frac{4}{m}} + \Omega A^{1-\frac{2}{m}} + \Theta A^{\frac{1}{m}} = 0$ holds, where ψ , Ω , and Θ are homogeneous polynomials in y . Then $\psi = \Omega = \Theta = 0$.*

6.6.2 Proof of Theorem 6.1.3

From (6.4.13), we obtain $\bar{G}^i y^j - \bar{G}^j y^i = G^i y^j - G^j y^i + H^{ij}$, where

$$\begin{cases} H^{ij} = \left(\frac{-(\alpha+\gamma)}{1+2s-\rho}\right)\left(s_0^i y^j - s_0^j y^i\right) + \left[\frac{(1+2s-\rho)r_{00}+2(\alpha+\gamma)s_0}{\lambda(1+s)^3(1+2s-\rho)}\right] \\ \quad \left\{b_r - s\left(\frac{Y_r}{\alpha} + d_r\right)\right\}(b^i y^j - b^j y^i). \end{cases}$$

In view of the above definition, if F and $\tilde{F}$ are Douglas metrics, then H^{ij} is a homogeneous polynomial of degree three in y . This theorem enables us to obtain several new Douglas metrics from a single one.

CHAPTER 7

Curvatures on Homogeneous Finsler Space with Finsler Metrics

The present chapter examines the curvature characteristics of homogeneous Finsler spaces admitting the exponential (α, β) -metric defined as (1.4.6). An explicit closed-form expression for the S -curvature is established within the framework of homogeneous Finsler spaces with exponential (α, β) -metric. Moreover, by exploiting the structural relationship between S -curvature and average Berwald curvature, a complete characterization of the average Berwald curvature is obtained for homogeneous Finsler spaces admitting exponential (α, β) -metric as given in (1.4.6).

7.1 S -curvature Homogeneous Exponential (α, β) -Metric

Definition 7.1.1. *The Busemann-Hausdorff volume form, $dV_{BH} = \sigma_{BH}(x)dx$ is defined as [38]:*

$$\sigma_{BH} = \frac{Vol(B^n)}{Vol\{(y^i) \in R^n | F(x, y^i e^i) < 1\}},$$

The Holmes-Thompson volume form, $dV_{HT} = \sigma_{HT}(x)dx$ is defined as:

$$\sigma_{HT}(x) = \frac{1}{Vol B^n} \int_{\{y^i \in R^n | F(x, y^i e^i) < 1\}} \det(g_{ij}) dy.$$

Both volume forms coincide in the case of Riemannian metric, i.e., $dV_{BH} = dV_{HT} = \sqrt{\det g_{ij}(x)} dx$.

Definition 7.1.2. *A Finsler space $(G/H, F)$ is characterized as having nearly isotropic S -*

curvature if there exists a smooth function $d(x)$ defined on M and a closed 1-form η that fulfill the following conditions [46]:

$$S(x, y) = (\eta + 1)(d(x)F(y) + \eta(y)), \quad x \in G/H, \quad y \in T_x M.$$

If $\eta(y) = 0$, then $(G/H, F)$ is said to have isotropic S -curvature.

And if $\eta(y) = 0$, $c(x) = \text{constant}$, then $(G/H, F)$ is said to have constant S -curvature [157].

In 2009, Shen and Cheng [38] derived the formula for the S -curvature of an (α, β) -metric within a local coordinate system, as follows:

$$S = \left(2\psi - \frac{f'(b)}{bf(b)} \right) (r_0 + s_0) - \alpha^{-1} \frac{\Phi}{2\Delta^2} (r_{00} - 2\alpha Q s_0), \quad (7.1.1)$$

where,

$$\begin{cases} Q = \frac{\phi'}{\phi - u\phi'}, & \psi = \frac{Q'}{2\Delta}, & \Delta = 1 + uQ + (b^2 - u^2)Q', \\ \Phi = (uQ' - Q)(1 + n\Delta + uQ) - (b^2 - u^2)(1 + uQ)Q''. \end{cases}$$

Here, $f(b)$ appearing in equation (7.1.1) is defined as

$$f(b) = \begin{cases} \frac{\int_0^\pi \sin^{n-2} t dt}{\int_0^\pi \frac{\sin^{n-2} t}{\phi(b \cos t)^n} dt} & \text{if } dV = dV_{BH}, \\ \frac{\int_0^\pi (\sin^{n-2} t) T(b \cos t) dt}{\int_0^\pi (\sin^{n-2} t) dt} & \text{if } dV = dV_{HT}. \end{cases}$$

Here, dV_{BH} and dV_{HT} denote the Busemann–Hausdorff volume form and the Holmes–Thompson volume form, respectively, as defined in Definition (7.1.1). Furthermore, when the Riemannian length b is constant, the sum $r_0 + s_0$ vanishes, and equation (7.1.1) reduces

to

$$S = \alpha^{-1} \frac{\Phi}{2\Delta^2} (r_{00} - 2\alpha Q_{s_0}).$$

We now recall the formula for the S -curvature in a homogeneous Finsler space, as established in [118].

Theorem 7.1.3. *Let $F = \alpha\phi(u)$ be a G -invariant (α, β) -metric on the reductive homogeneous Finsler space G/H , where the Lie algebra $\mathfrak{g}$ has a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then the S -curvature is given by*

$$S(H, y) = \frac{\Phi}{2\alpha\Delta^2} (< [v, y]_m, y > + \alpha Q < [v, y]_m, v >), \quad (7.1.2)$$

where $v \in \mathfrak{m}$ corresponds to the 1-form β , and $\mathfrak{m}$ is identified with the tangent space $T_H(G/H)$ at the point H of the homogeneous Finsler space G/H .

By applying Theorem (7.1.3), we obtain the S -curvature formula for the homogeneous exponential (α, β) -metric.

Theorem 7.1.4. *Consider a reductive homogeneous Finsler space G/H , where the Lie algebra decomposes as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Let $F = \alpha e^{\beta/\alpha} + \beta e^{-\beta/\alpha}$ be a G -invariant exponential (α, β) -metric on G/H . Then the S -curvature of F is given by*

$$S(H, y) = \frac{\xi_1}{\xi_2} \left(\frac{< [v, y]_m, y >}{\alpha} + \frac{e^u + e^{-u} - ue^{-u}}{e^u - ue^u + u^2 e^{-u}} (< [v, y]_m, v >) \right). \quad (7.1.3)$$

Where,

$$\begin{aligned} \xi_1 = & \left\{ (2ue^{2u} - 3u^2e^{-2u} + 2u^3e^{-2u} - e^{2u} - 1) \right. \\ & \times (e^{2u} - 2ue^{2u} + u^2e^{2u} + 2u^2 - 2u^3 + u^4e^{-2u} \\ & + \eta(e^{2u} - 2u^2e^{2u} + 2u + 4u^3e^{-2u} - 2u^4e^{-2u} \\ & + b^2(e^{2u} + 2u - 2 - 2ue^{-2u} + u^2e^{-2u}))) \} \\ & - [(b^2(e^u + ue^{-u}) - u^2e^u - u^3e^{-u}) \end{aligned}$$

$$\begin{aligned}
& \times (2e^{3u} + 2e^u - 2e^{-u} + 16ue^{-u} - 16u^2e^{-u} - 10ue^u \\
& - 6u^2e^{-u} + 8u^3e^{-u} + 8u^2e^u + 6u^2e^{-3u} - 2u^3e^{-3u}) \Big], \\
\xi_2 &= 2 \left(e^{2u} - 2u^2e^{2u} + 2u + 4u^3e^{-2u} - 2u^4e^{-2u} + b^2(e^{2u} + 2u - 2 - 2ue^{-2u} + u^2e^{-2u}) \right)^2.
\end{aligned}$$

Proof. To compute the S -curvature of a G -invariant exponential (α, β) -metric on a reductive homogeneous Finsler space G/H , we first determine the parameters required in Theorem (7.1.3).

Recall from [118] that

$$\begin{cases} Q = \frac{\phi'}{\phi - s\phi'}, \\ \Delta = 1 + uQ + (b^2 - u^2)Q', \\ \Phi = - \left(Q - uQ' \right) (1 + n\Delta + uQ) - (b^2 - u^2)(1 + uQ)Q''. \end{cases}$$

We now compute these parameters for the exponential (α, β) -metric.

$$\begin{cases} Q = \frac{e^u + e^{-u} - ue^{-u}}{e^u - ue^u + u^2e^{-u}}, \\ Q' = \frac{e^{2u} + 2u - 2 - 2ue^{-2u} + u^2e^{-2u}}{e^u - ue^u + u^2e^{-u}}, \\ Q'' = \frac{A_1}{A_2}, \end{cases}$$

where,

$$A_1 = (e^u - ue^u + u^2e^{-u})(2e^{2u} + 2 - 2e^{-2u} + 6ue^{-2u} - 2u^2e^{-2u}) + 2u(e^u - 2e^{-u} + ue^{-u})(e^{2u} + 2u - 2 - 2ue^{-2u} + u^2e^{-2u}),$$

$$A_2 = (e^u - ue^u + u^2e^{-u})^3,$$

$$\Delta = \frac{B_1}{B_2},$$

$$B_1 = (e^u - ue^u + u^2e^{-u})^2 + u(e^u + e^{-u} - ue^{-u})(e^u - ue^u + u^2e^{-u}) + (b^2 - u^2)(e^{2u} + 2u - 2 - 2ue^{-2u} + u^2e^{-2u}),$$

$$B_2 = (e^u - ue^u + u^2e^{-u})^2,$$

$$\Phi = \frac{1}{(e^u - ue^u + u^2e^{-u})^4} \left\{ \left[\{ u(e^{2u} + 2u - 2 - 2ue^{-2u} + u^2e^{-2u}) - (e^u + e^{-u} - ue^{-u})(e^u - ue^u + u^2e^{-u}) \} \times \{ (e^u - ue^u + u^2e^{-u})^2 + \eta((e^u - ue^u + u^2e^{-u})^2 + u(e^u + e^{-u} - ue^{-u})(e^u - ue^u + u^2e^{-u})) \} \right] \right\}$$

$$\begin{aligned}
& ue^u + u^2e^{-u}) + (b^2 - u^2) \times (e^{2u} + 2u - 2 - 2ue^{-2u} + u^2e^{-2u})) + u(e^u + e^{-u} - ue^{-u}) \\
& (e^u - ue^u + u^2e^{-u}) \} \Big] - [(b^2 - u^2)\{(e^u - ue^u + u^2e^{-u}) + u(e^u + e^{-u} - ue^{-u})\}\{(e^u - ue^u + \\
& u^2e^{-u})(2e^{2u} + 2 - 2e^{-2u} + 6ue^{-2u} - 2u^2e^{-2u}) + 2u(e^u - 2e^{-u} + ue^{-u})(e^{2u} + 2u - 2 - 2ue^{-2u} + \\
& u^2e^{-2u})\}\} \Big].
\end{aligned}$$

Substituting these values into equation (7.1.1), we obtain

$$S(H, y) = \frac{\xi_1}{\xi_2} \left(\frac{\langle [v, y]_m, y \rangle}{\alpha} + \frac{e^u + e^{-u} - ue^{-u}}{e^u - ue^u + u^2e^{-u}} \langle [v, y]_m, v \rangle \right). \quad (7.1.4)$$

Where,

$$\begin{aligned}
\xi_1 = & \left[\{ (2ue^{2u} - 3u^2e^{-2u} + 2u^3e^{-2u} - e^{2u} - 1) \right. \\
& \times (e^{2u} - 2ue^{2u} + u^2e^{2u} + 2u^2 - 2u^3 + u^4e^{-2u} \\
& + \eta(e^{2u} - 2u^2e^{2u} + 2u + 4u^3e^{-2u} - 2u^4e^{-2u} \\
& + b^2(e^{2u} + 2u - 2 - 2ue^{-2u} + u^2e^{-2u}))) \} \\
& - [(b^2(e^u + ue^{-u}) - u^2e^u - u^3e^{-u}) \\
& \times (2e^{3u} + 2e^u - 2e^{-u} + 16ue^{-u} - 16u^2e^{-u} - 10ue^u \\
& - 6u^2e^{-u} + 8u^3e^{-u} + 8u^2e^u + 6u^2e^{-3u} - 2u^3e^{-3u}))] \Big], \\
\xi_2 = & 2(e^{2u} - 2u^2e^{2u} + 2u + 4u^3e^{-2u} - 2u^4e^{-2u} + b^2(e^{2u} + 2u - 2 - 2ue^{-2u} + u^2e^{-2u}))^2.
\end{aligned}$$

Where $v \in \mathfrak{m}$ corresponds to the 1-form β , and $\mathfrak{m}$ is identified with the tangent space $T_H(G/H)$ at the identity coset H . This completes the proof of Theorem (7.1.4). As an immediate consequence, we have the following corollary. $\square$

Corollary 7.1.5. *Let $(G/H, F)$ be a homogeneous exponential (α, β) -metric under the same conditions as in Theorem (7.1.4). Then $(G/H, F)$ has isotropic S -curvature if and only if $S = 0$.*

Proof. Recall from Definition (7.1.2) that if $(G/H, F)$ has isotropic S -curvature, then

$$S(x, y) = (n+1)(d(x)F(y) + \eta(y)), \quad x \in G/H, \quad y \in T_x M.$$

□

For a homogeneous Finsler space, it suffices to compute the S -curvature at the origin, that is, at $x = H$ and $y = v$. Substituting $x = H$ and $y = v$ into the S -curvature formula of Theorem (7.1.4), we obtain $d(H) = 0$, which implies $S(H, y) = 0$ for all $y \in T_x M$. Hence, G/H has vanishing S -curvature.

7.2 Mean Berwald Curvature

This section is devoted to computing the mean Berwald curvature of a homogeneous Finsler space equipped with an exponential (α, β) -metric.

We first recall the notion of mean Berwald curvature for Finsler spaces. In [39], the author introduced the mean Berwald curvature for a Finsler space as follows. Consider $E_{ij} = \frac{1}{2} \frac{\partial^2 S(x, y)}{\partial y^i \partial y^j} = \frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j} \left(\frac{\partial G^m}{\partial y^m} \right) (x, y)$. Consider the symmetric bilinear form $E_y : T_x M \times T_x M \rightarrow \mathbb{R}$, defined by

$$E_y(a, b) = E_{ij} a^i b^j,$$

where $a = a^i \frac{\partial}{\partial y^i}$, $b = b^j \frac{\partial}{\partial y^j} \in T_x M$, $x \in M$.

Then E -tensor defined as family of symmetric forms $\{E_y\}$ can be seen as [118]:

$$E = \{E_y : y \in TM / \{0\}\},$$

which is known as E -Curvature or mean Berwald curvature.

Before stating the main result of this section, we establish several key quantities are re-

quired in the sequel. At the origin,

$$\left\{ \begin{array}{l} a_{ij} = \delta_i^j, y_i = y^i, \\ \alpha_{y^i} = \frac{y_i}{\alpha}, \beta_{y^i} = b_i, \\ u_{y^i} = \frac{\partial}{\partial y^i} \left(\frac{\beta}{\alpha} \right) = \frac{\alpha b_i - u y_i}{\alpha^2}, \\ u_{y^i y^j} = \frac{\partial}{\partial y^j} \left(\frac{\alpha b_i - u y_i}{\alpha^2} \right) \\ = \frac{3u y_i y_j - (b_i y_j + b_j y_i) \alpha - \alpha^2 u \delta_j^i}{\alpha^4}. \end{array} \right.$$

Consider $\Omega = \frac{\xi_1}{\xi_2}$.

$$\begin{aligned} \xi_1 = & \left[\{ (2ue^{2u} - 3u^2e^{-2u} + 2u^3e^{-2u} - e^{2u} - 1) \right. \\ & \times (e^{2u} - 2ue^{2u} + u^2e^{2u} + 2u^2 - 2u^3 + u^4e^{-2u} \\ & + \eta(e^{2u} - 2u^2e^{2u} + 2u + 4u^3e^{-2u} - 2u^4e^{-2u} \\ & + b^2(e^{2u} + 2u - 2 - 2ue^{-2u} + u^2e^{-2u}))) \} \\ & - [(b^2(e^u + ue^{-u}) - u^2e^u - u^3e^{-u}) \\ & \times (2e^{3u} + 2e^u - 2e^{-u} + 16ue^{-u} - 16u^2e^{-u} - 10ue^u \\ & - 6u^2e^{-u} + 8u^3e^{-u} + 8u^2e^u + 6u^2e^{-3u} - 2u^3e^{-3u})] \Big], \end{aligned}$$

$$\xi_2 = 2(e^{2u} - 2u^2e^{2u} + 2u + 4u^3e^{-2u} - 2u^4e^{-2u} + b^2(e^{2u} + 2u - 2 - 2ue^{-2u} + u^2e^{-2u}))^2.$$

$$\frac{\partial \Omega}{\partial y^j} = \frac{K_1}{K_2} u_{y^j}, \quad (7.2.1)$$

and

$$\frac{\partial^2 \Omega}{\partial y^j \partial y^i} = \frac{K_1}{K_2} u_{y^j y^i} + \frac{K_3}{K_4} u_{y^j} u_{y^i}, \quad (7.2.2)$$

where,

$$\left\{ \begin{aligned} K_1 = & \left[\left\{ (4ue^{2u} - 6ue^{-2u} + 12u^2e^{-2u} - 4u^3e^{-2u}) \left\{ e^{2u} - 2ue^{2u} + u^2e^{2u} + 2u^2 \right. \right. \right. \\ & - 2u^3 + u^4e^{-2u} + \eta((e^{2u} - 2u^2e^{2u} + 2u + 4u^3e^{-2u} - 2u^4e^{-2u} + b^2(e^{2u} + 2u \\ & - 2 - 2ue^{-2u} + u^2e^{-2u})) \left. \right\} + (2ue^{2u} - 3u^2e^{-2u} + 2u^3e^{-2u} - e^{2u} - 1) \left(-2ue^{2u} \right. \\ & + 2u^2e^{2u} + 4u - 6u^2 + 4u^3e^{-2u} - 2u^4e^{-2u} + \eta(2e^{2u} - 4ue^{2u} - 4u^2e^{2u} + 2 + \\ & 12u^2e^{-2u} + 16u^3e^{-2u} + 4u^4e^{-2u} + b^2(2e^{2u} + 2 - 2e^{-2u} + 6ue^{-2u} - 2u^2e^{-2u}) \\ & \left. \left. \left. \right) \right) \right\} - \left\{ \left(b^2(e^u + e^{-u} - ue^{-u}) + (-2ue^u - u^2e^u - 3u^2e^{-u} + u^3e^{-u}) \right) \left(2e^{3u} \right. \right. \right. \\ & + 2e^u - 2e^{-u} + 16ue^{-u} - 16u^2e^{-u} - 10ue^u - 6u^2e^{-u} + 8u^3e^{-u} + 8u^2e^u + 6u^2 \\ & e^{-3u} - 2u^3e^{-3u} \left. \right) + \left(b^2(e^u + ue^{-u}) + (-u^2e^u - u^3e^{-u}) \right) \left(6e^{3u} - 8e^u + 18e^{-u} \right. \\ & - 60ue^{-u} + 46u^2e^{-u} + 6ue^u - 8u^3e^{-u} + 8u^2e^u + 12ue^{-3u} - 24u^2e^{-3u} + \\ & 6u^3e^{-3u} \left. \right) \left. \right\} \right] + \left\{ e^{2u} - 2u^2e^{2u} + 2u + 4u^3e^{-2u} - 2u^4e^{-2u} + b^2(e^{2u} + 2u - 2 \right. \\ & - 2ue^{-2u} + u^2e^{-2u}) \left. \right\} - 4 \left[\left(2e^{2u} - 4ue^{2u} - 4u^2e^{2u} + 2 + 12u^2e^{-2u} \right. \right. \\ & - 16u^3e^{-2u} + 4u^4e^{-2u} + b^2(2e^{2u} + 2 - 2e^{-2u} + 6ue^{-2u} - 2u^2e^{-2u}) \left. \right) \\ & \left(\left\{ (2ue^{2u} - 3u^2e^{-2u} + 2u^3e^{-2u} - e^{2u} - 1) \left\{ e^{2u} - 2ue^{2u} + u^2e^{2u} + 2u^2 \right. \right. \right. \right. \\ & - 2u^3 + u^4e^{-2u} + \eta(e^{2u} - 2u^2e^{2u} + 2u + 4u^3e^{-2u} - 2u^4e^{-2u} + b^2 \\ & (e^{2u} + 2u - 2 - 2ue^{-2u} + u^2e^{-2u})) \left. \right\} - \left((b^2(e^u + ue^{-u}) + (-u^2e^u \right. \\ & - u^3e^{-u})) (2e^{3u} + 2e^u - 2e^{-u} + 16ue^{-u} - 16u^2e^{-u} - 10ue^u - 6u^2e^{-u} + \\ & 8u^3e^{-u} + 8u^2e^u + 6u^2e^{-3u} - 2u^3e^{-3u}) \left. \right) \right] \left. \right\}. \end{aligned} \right.$$

$$\left\{ K_2 = 4 \left\{ e^{2u} - 2u^2e^{2u} + 2u + 4u^3e^{-2u} - 2u^4e^{-2u} + b^2(e^{2u} + 2u - 2 - 2ue^{-2u} + u^2e^{-2u}) \right\}^3, \right.$$

$$\begin{aligned}
K_3 = & \left[(4e^{2u} + 8ue^{2u} - 6e^{-2u} + 36ue^{-2u} - 36u^2e^{-2u} + 8u^3e^{-2u}) \left\{ e^{2u} - 2ue^{2u} \right. \right. \\
& + u^2e^{2u} + 2u^2 - 2u^3 + u^4e^{-2u} + \eta(e^{2u} - 2u^2e^{2u} + 2u + 4u^3e^{-2u} - 2u^4e^{-2u} + \\
& \left. \left. b^2(e^{2u} + 2u - 2 - 2ue^{-2u} + u^2e^{-2u})) \right\} + (4ue^{2u} - 6ue^{-2u} + 12u^2e^{-2u} - 4u^3 \right. \\
& e^{-2u})(-2ue^{2u} + 2u^2e^{2u} + 4u - 6u^2 + 4u^3e^{-2u} - 2u^4e^{-2u} + \eta(2e^{2u} - 4ue^{2u} \\
& - 4u^2e^{2u} + 2 + 12u^2e^{-2u} - 16u^3e^{-2u} + 4u^4e^{-2u} + b^2(2e^{2u} + 2 - 2e^{-2u} + 6ue^{-2u} \\
& - 2u^2e^{-2u})) + (2e^{2u} + 4ue^{2u} - 6ue^{-2u} + 12u^2e^{-2u} - 4u^3e^{-2u} - 2e^{2u})(-2ue^{2u} + \\
& 2u^2e^{2u} + 4u - 6u^2 + 4u^3e^{-2u} - 2u^4e^{-2u} + \eta(2e^{2u} - 4ue^{2u} - 4u^2e^{2u} + 2 + 12u^2 \\
& e^{-2u} + 16u^3e^{-2u} + 4u^4e^{-2u} + b^2(2e^{2u} + 2 - 2e^{-2u} + 6ue^{-2u} - 2u^2e^{-2u})) + (2ue^{2u} \\
& - 3u^2e^{-2u} + 2u^3e^{-2u} - e^{2u} - 1)(-2e^{2u} + 4u^2e^{2u} + 4 - 12u + 12u^2e^{2u} - 16u^3e^{-2u} \\
& + 4u^4e^{-2u}) + \eta(-16ue^{2u} - 8u^2e^{2u} + 24e^{-2u} + 24u^2e^{-2u} - 16u^3e^{-2u} - 8u^4e^{-2u} \\
& + b^2(4e^{2u} + 10e^{-2u} - 16ue^{-2u} + 4u^2e^{-2u})) - \left\{ (b^2(e^u - 2e^{-u} + ue^{-u}) + (-2e^u \right. \\
& - 4ue^u - u^2e^u - 6ue^{-u} + 6u^2e^{-u} - u^3e^{-u}))(2e^{3u} + 2e^u - 2e^{-u} + 16ue^{-u} - 16u^2e^{-u} \\
& - 10ue^u - 6u^2e^{-u} + 8u^3e^{-u} + 8u^2e^u + 6u^2e^{-3u} - 2u^3e^{-3u}) + (b^2(e^u + e^{-u} \\
& - ue^{-u}) + (-2ue^u - u^2e^u - 3u^2e^{-u} + u^3e^{-u}))(6e^{3u} + 2e^u + 18e^{-u} \\
& - 60ue^{-u} + 22u^2e^{-u} - 10e^u - 10ue^u + 24u^2e^{-u} - 8u^3e^{-u} + 16ue^u \\
& + 8u^2e^u + 12ue^{-3u} - 24u^2e^{-3u} + 6u^3e^{-3u}) + (b^2(e^u + e^{-u} - ue^{-u}) \\
& + (-2ue^u - u^2e^u - 3u^2e^{-u} + u^3e^{-u}))(6e^{3u} - 8e^u + 18e^{-u} - 60ue^{-u} + \\
& \left. \left. 46u^2e^{-u} + 6ue^u - 8u^3e^{-u} + 8u^2e^u + 12ue^{-3u} - 24u^2e^{-3u} + 6u^3e^{-3u}) \right\} \right] \\
& 2 \left\{ e^{2u} - 2u^2e^{2u} + 2u + 4u^3e^{-2u} - 2u^4e^{-2u} + b^2(e^{2u} + 2u - 2 - 2ue^{-2u} \right. \\
& \left. + u^2e^{-2u}) \right\} + 2 \left(2e^{2u} - 4ue^{2u} - 4u^2e^{2u} + 2 + 12u^2e^{-2u} - 16u^3e^{-2u} + \right. \\
& \left. 4u^4e^{-2u} + b^2(2e^{2u} + 2 - 2e^{-2u} + 6ue^{-2u} - 2u^2e^{-2u}) \right) \left[\left\{ (4ue^{2u} - 6ue^{-2u} \right. \right. \\
& + 12u^2e^{-2u} - 4u^3e^{-2u}) \{ e^{2u} - 2ue^{2u} + u^2e^{2u} + 2u^2 - 2u^3 + u^4e^{-2u} + \eta \\
& (e^{2u} - 2u^2e^{2u} + 2u + 4u^3e^{-2u} - 2u^4e^{-2u} + b^2(e^{2u} + 2u - 2 - 2ue^{-2u} \\
& + u^2e^{-2u})) \} + (2ue^{2u} - 3u^2e^{-2u} + 2u^3e^{-2u} - e^{2u} - 1)(-2ue^{2u} + 2u^2e^{2u} \\
& + 4u - 6u^2 + 4u^3e^{-2u} - 2u^4e^{-2u} + \eta(2e^{2u} - 4ue^{2u} - 4u^2e^{2u} + 2 + 12u^2e^{-2u} \\
& + 16u^3e^{-2u} + 4u^4e^{-2u} + b^2(2e^{2u} + 2 - 2e^{-2u} + 6ue^{-2u} - 2u^2e^{-2u})) \} \right]
\end{aligned}$$

$$\left(- \left\{ (b^2(e^u + e^{-u} - ue^{-u}) + (-2ue^u - u^2e^u - 3u^2e^{-u} + u^3e^{-u}))(2e^{3u} \right. \right. \\
+ 2e^u - 2e^{-u} + 16ue^{-u} - 16u^2e^{-u} - 10ue^u - 6u^2e^{-u} + 8u^3e^{-u} + 8u^2e^u + \\
6u^2e^{-3u} - 2u^3e^{-3u}) + (b^2(e^u + ue^{-u}) + (-u^2e^u - u^3e^{-u}))(6e^{3u} - 8e^u \\
+ 18e^{-u} - 60ue^{-u} + 46u^2e^{-u} + 6ue^u - 8u^3e^{-u} + 8u^2e^u + 12ue^{-3u} \\
- 24u^2e^{-3u} + 6u^3e^{-3u}) \left. \right\} - \left[-4(-16ue^{2u} - 8u^2e^{2u} + 24ue^{-2u} - \right. \\
72u^2e^{-2u} + 48u^3e^{-2u} - 8u^4e^{-2u} + b^2(4e^{2u} + 10e^{-2u} - 16ue^{-2u} + \\
4u^2e^{-2u})) \left(\{ (2ue^{2u} - 3u^2e^{-2u} + 2u^3e^{-2u} - e^{2u} - 1) \{ e^{2u} - 2ue^{2u} + u^2e^{2u} \right. \right. \\
+ 2u^2 - 2u^3 + u^4e^{-2u} + \eta(e^{2u} - 2u^2e^{2u} + 2u + 4u^3e^{-2u} - 2u^4e^{-2u} + b^2 \\
(e^{2u} + 2u - 2 - 2ue^{-2u} + u^2e^{-2u})) \} - \left((b^2(e^u + ue^{-u}) + (-u^2e^u - u^3e^{-u})) \right. \\
(2e^{3u} + 2e^u - 2e^{-u} + 16ue^{-u} - 16u^2e^{-u} - 10ue^u - 6u^2e^{-u} + 8u^3e^{-u} + 8u^2e^u + \\
6u^2e^{-3u} - 2u^3e^{-3u}) \left. \right) + \left(2e^{2u} - 4ue^{2u} - 4u^2e^{2u} + 2 + 12u^2e^{-2u} - 16u^3e^{-2u} + \right. \\
4u^4e^{-2u} + b^2(2e^{2u} + 2 - 2e^{-2u} + 6ue^{-2u} - 2u^2e^{-2u})) \left. \right\} \left\{ ((4ue^{2u} - 6ue^{-2u} + 12u^2e^{-2u} \right. \\
- 4u^3e^{-2u}) \{ e^{2u} - 2ue^{2u} + u^2e^{2u} + 2u^2 - 2u^3 + u^4e^{-2u} + \eta(e^{2u} - 2u^2e^{2u} + 2u + 4u^3 \\
e^{-2u} - 2u^4e^{-2u} + b^2((e^{2u} + 2u - 2 - 2ue^{-2u} + u^2e^{-2u}))) \} + (2ue^{2u} - 3u^2e^{-2u} + 2u^3 \\
e^{-2u} - e^{2u} - 1) \left(-2ue^{2u} + 2u^2e^{2u} + 4u - 6u^2 + 4u^3e^{-2u} - 2u^4e^{-2u} + \eta(2e^{2u} - 4ue^{2u} \\
- 4u^2e^{2u} + 2 + 12u^2e^{-2u}) - 16u^3e^{-2u} + 4u^4e^{-2u} + b^2(2e^{2u} + 2 - 2e^{-2u} + 6ue^{-2u} \\
- 2u^2e^{-2u})) \right) - \left(b^2(e^u + e^{-u} - ue^{-u}) + (-2ue^u - u^2e^u - 3u^2e^{-u} + u^3e^{-u})(2e^{3u} \right. \\
+ 2e^u - 2e^{-u} + 16ue^{-u} - 16u^2e^{-u} - 10ue^u - 6u^2e^{-u} + 8u^3e^{-u} + 8u^2e^u + 6u^2e^{-3u} \\
- 2u^3e^{-3u}) + (b^2(e^u + ue^{-u}) + (-u^2e^u - u^3e^{-u}))(6e^{3u} + 2e^u + 18e^{-u} - 48ue^{-u} \\
+ 46u^2e^{-u} - 10e^u - 10ue^u - 12ue^{-u} - 8u^3e^{-u} + 16ue^u + 8u^2e^u + 12ue^{-3u} - 24u^2e^{-3u} \\
+ 6u^3e^{-3u}) \left. \right\} \left\{ 4 \{ e^{2u} - 2u^2e^{2u} + 2u + 4u^3e^{-2u} - 2u^4e^{-2u} + b^2(e^{2u} + 2u - 2 - 2ue^{-2u} \right. \\
+ u^2e^{-2u}) \}^3 \left. \right\} - \left[12 \{ e^{2u} - 2u^2e^{2u} + 2u + 4u^3e^{-2u} - 2u^4e^{-2u} + b^2(e^{2u} + 2u - 2 - 2ue^{-2u} \right. \\
+ u^2e^{-2u}) \}^2 \left(2e^{2u} - 4ue^{2u} - 4u^2e^{2u} + 2 + 12u^2e^{-2u} - 16u^3e^{-2u} + 4u^4e^{-2u} + b^2(2e^{2u} + 2 \right. \\
- 2e^{-2u} + 6ue^{-2u} - 2u^2e^{-2u})) \left. \right] \left[\left\{ (4ue^{2u} - 6ue^{-2u} + 12u^2e^{-2u} - 4u^3e^{-2u}) \{ e^{2u} - 2ue^{2u} + \right. \right. \\
u^2e^{2u} + 2u^2 - 2u^3 + u^4e^{-2u} + \eta(e^{2u} - 2u^2e^{2u} + 2u + 4u^3e^{-2u} - 2u^4e^{-2u} + b^2(e^{2u} + 2u \\
- 2 - 2ue^{-2u} + u^2e^{-2u})) \} \right.$$

$$\left\{ \begin{aligned}
& +(2ue^{2u} - 3u^2e^{-2u} + 2u^3e^{-2u} - e^{2u} - 1)(-2ue^{2u} + 2u^2e^{2u} \\
& + 4u - 6u^2 + 4u^3e^{-2u} - 2u^4e^{-2u} + \eta(2e^{2u} - 4ue^{2u} - 4u^2e^{2u} + 2 + 12u^2e^{-2u} + 16u^3e^{-2u} \\
& + 4u^4e^{-2u} + b^2(2e^{2u} + 2 - 2e^{-2u} + 6ue^{-2u} - 2u^2e^{-2u})) \Big\} - \Big\{ (b^2(e^u + e^{-u} - ue^{-u}) \\
& + (-2ue^u - u^2e^u - 3u^2e^{-u} + u^3e^{-u}))(2e^{3u} + 2e^u - 2e^{-u} + 16ue^{-u} - 16u^2e^{-u} - 10ue^u \\
& - 6u^2e^{-u} + 8u^3e^{-u} + 8u^2e^u + 6u^2e^{-3u} - 2u^3e^{-3u}) + (b^2(e^u + ue^{-u}) + (-u^2e^u - u^3e^{-u})) \\
& (6e^{3u} - 8e^u + 18e^{-u} - 60ue^{-u} + 46u^2e^{-u} + 6ue^u - 8u^3e^{-u} + 8u^2e^u + 12ue^{-3u} - 24u^2e^{-3u} \\
& + 6u^3e^{-3u}) \Big\} \Big] + 2 \Big\{ e^{2u} - 2u^2e^{2u} + 2u + 4u^3e^{-2u} - 2u^4e^{-2u} + b^2(e^{2u} + 2u - 2 - 2ue^{-2u} \\
& + u^2e^{-2u}) \Big\} - 4 \Big[(2e^{2u} - 4ue^{2u} - 4u^2e^{2u} + 2 + 12u^2e^{-2u} - 16u^3e^{-2u} + 4u^4e^{-2u} + b^2(2e^{2u} \\
& + 2 - 2e^{-2u} + 6ue^{-2u} - 2u^2e^{-2u})) \Big(\{ (2ue^{2u} - 3u^2e^{-2u} + 2u^3e^{-2u} - e^{2u} - 1) \\
& \{ e^{2u} - 2ue^{2u} + u^2e^{2u} + 2u^2 - 2u^3 + u^4e^{-2u} + \eta(e^{2u} - 2u^2e^{2u} + 2u + 4u^3e^{-2u} - 2u^4e^{-2u} + \\
& b^2(e^{2u} + 2u - 2 - 2ue^{-2u} + u^2e^{-2u})) \} - \Big((b^2(e^u + ue^{-u}) + (-u^2e^u - u^3e^{-u})) \\
& (2e^{3u} + 2e^u - 2e^{-u} + 16ue^{-u} - 16u^2e^{-u} - 10ue^u - 6u^2e^{-u} + 8u^3e^{-u} \\
& + 8u^2e^u + 6u^2e^{-3u} - 2u^3e^{-3u}) \Big) \Big] \Big].
\end{aligned} \right.$$

$$\left\{ K_4 = 16 \left\{ e^{2u} - 2u^2e^{2u} + 2u + 4u^3e^{-2u} - 2u^4e^{-2u} + b^2(e^{2u} + 2u - 2 - 2ue^{-2u} + u^2e^{-2u}) \right\}^6 \right.$$

Theorem 7.2.1. Let $(G/H, F)$ be a reductive homogeneous Finsler space with Lie algebra decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, and let $F = \alpha e^{\beta/\alpha} + \beta e^{-\beta/\alpha}$ be a G -invariant exponential

(α, β) -metric on G/H . Then the mean Berwald curvature is given by

$$\left\{ \begin{aligned} E_{i,j}(H, y) &= \frac{1}{2} \left[\frac{\partial}{\partial y^i} \left(\frac{1}{\alpha} \frac{\partial \Omega}{\partial y^j} - \frac{\Omega}{\alpha^2} \frac{y_j}{\alpha} \right) \langle [v, y]_m, y \rangle + \frac{\Omega}{\alpha} \right. \\ &(\langle [v, v_j]_m, y \rangle + \langle [v, y]_m, v_j \rangle) = \left(\frac{1}{\alpha} \Omega_{ij} - \frac{y_i}{\alpha^3} \Omega_j - \frac{y_j}{\alpha^3} \Omega_i \right. \\ &- \frac{\Omega}{\alpha^3} \delta_i^j + \frac{3\Omega}{\alpha^5} y_i y_j) \langle [v, y]_m, y \rangle + \left(\frac{1}{\alpha} \Omega_j - \frac{\Omega y_j}{\alpha^3} \right) \\ &(\langle [v, v_i]_m, y \rangle + \langle [v, y]_m, v_i \rangle) + \left(\frac{1}{\alpha} \Omega_i - \frac{\Omega y_i}{\alpha^3} \right) \\ &(\langle [v, v_j]_m, y \rangle + \langle [v, y]_m, v_j \rangle) + \frac{\Omega}{\alpha} (\langle [v, v_j]_m, v_i \rangle \\ &+ \langle [v, v_i]_m, v_j \rangle) + \left\{ \left(\frac{e^u + e^{-u} - u e^{-u}}{e^u - u e^u + u^2 e^{-u}} \right) \Omega_{ij} \right. \\ &+ \left(\frac{e^{2u} + 2u - 2 - 2u e^{-2u} + u^2 e^{-2u}}{(e^u - u e^u + u^2 e^{-u})^2} \right) \Omega_j u_{y^i} + \Omega_i \\ &\left(\frac{e^{2u} + 2u - 2 - 2u e^{-2u} + u^2 e^{-2u}}{(e^u - u e^u + u^2 e^{-u})^2} \right) u_{y^j} + \Omega u_{y^i y^j} \\ &\left. \left(\frac{e^{2u} + 2u - 2 - 2u e^{-2u} + u^2 e^{-2u}}{(e^u - u e^u + u^2 e^{-u})^2} \right) + \Omega u_{y^j} u_{y^i} \frac{A_1}{A_2} \right\} \langle [v, y]_m, v \rangle \\ &+ \left\{ \left(\frac{e^u + e^{-u} - u e^{-u}}{e^u - u e^u + u^2 e^{-u}} \right) \Omega_i + \left(\frac{e^{2u} + 2u - 2 - 2u e^{-2u} + u^2 e^{-2u}}{(e^u - u e^u + u^2 e^{-u})^2} \right) \right. \\ &\left. u_{y^i} \Omega \right\} \langle [v, v_j]_m, v \rangle + \left\{ \left(\frac{e^u + e^{-u} - u e^{-u}}{e^u - u e^u + u^2 e^{-u}} \right) \Omega_j \right. \\ &+ \Omega \left(\frac{e^{2u} + 2u - 2 - 2u e^{-2u} + u^2 e^{-2u}}{(e^u - u e^u + u^2 e^{-u})^2} \right) u_{y^j} \left. \right\} \\ &\left. \langle [v, v_i]_m, v \rangle \right]. \end{aligned} \right. \quad (7.2.3)$$

Proof. From Theorem (7.1.3), recall that the S -curvature at the origin given by equation (7.1.4) is as follows:

$$\frac{\xi_1}{\xi_2} \left(\frac{\langle [v, y]_m, y \rangle}{\alpha} - \frac{e^u + e^{-u} - u e^{-u}}{e^u - u e^u + u^2 e^{-u}} (\langle [v, y]_m, v \rangle) \right).$$

Where,

$$\begin{aligned} \xi_1 &= \left[\{ (2u e^{2u} - 3u^2 e^{-2u} + 2u^3 e^{-2u} - e^{2u} - 1) \right. \\ &\times (e^{2u} - 2u e^{2u} + u^2 e^{2u} + 2u^2 - 2u^3 + u^4 e^{-2u}) \\ &\left. + \eta (e^{2u} - 2u^2 e^{2u} + 2u + 4u^3 e^{-2u} - 2u^4 e^{-2u}) \right]
 \end{aligned}$$

$$\begin{aligned}
& + b^2(e^{2u} + 2u - 2 - 2ue^{-2u} + u^2e^{-2u})) \} \\
& - [(b^2(e^u + ue^{-u}) - u^2e^u - u^3e^{-u}) \\
& \times (2e^{3u} + 2e^u - 2e^{-u} + 16ue^{-u} - 16u^2e^{-u} - 10ue^u \\
& - 6u^2e^{-u} + 8u^3e^{-u} + 8u^2e^u + 6u^2e^{-3u} - 2u^3e^{-3u}))],
\end{aligned}$$

$$\xi_2 = 2(e^{2u} - 2u^2e^{2u} + 2u + 4u^3e^{-2u} - 2u^4e^{-2u} + b^2(e^{2u} + 2u - 2 - 2ue^{-2u} + u^2e^{-2u}))^2.$$

Let $\Omega = \frac{\xi_1}{\xi_2}$, which gives

$$S(H, y) = \frac{\Omega}{\alpha} \langle [v, y]_m, y \rangle - \Omega \frac{e^u + e^{-u} - ue^{-u}}{e^u - ue^u + u^2e^{-u}} \langle [v, y]_m, v \rangle.$$

Consider $S(H, y) = A + B$, where $A = \frac{\Omega}{\alpha} \langle [v, y]_m, y \rangle$, $B = \Omega \left(\frac{e^u + e^{-u} - ue^{-u}}{e^u - ue^u + u^2e^{-u}} \right) \langle [v, y]_m, v \rangle$

Now, by definition of mean Berwald curvature

$$E_{ij} = \frac{1}{2} \frac{\partial^2 S(H, y)}{\partial y^j \partial y^i} = \frac{1}{2} \left(\frac{\partial^2 A}{\partial y^i \partial y^j} + \frac{\partial^2 B}{\partial y^i \partial y^j} \right). \quad (7.2.4)$$

Next, we calculate $\frac{\partial^2 A}{\partial y^i \partial y^j}$ and $\frac{\partial^2 B}{\partial y^i \partial y^j}$ as follows:

$$\begin{aligned}
\frac{\partial A}{\partial y^j} &= \frac{\partial}{\partial y^j} \left(\frac{\Omega}{\alpha} \langle [v, y]_m, y \rangle \right) = \left(\frac{1}{\alpha} \frac{\partial \Omega}{\partial y^j} - \frac{\Omega}{\alpha^2} \frac{y_j}{\alpha} \right) \langle [v, y]_m, y \rangle + \frac{\Omega}{\alpha} (\langle [v, v_j]_m, y \rangle \\
&+ \langle [v, y]_m, v_j \rangle),
\end{aligned}$$

$$\left\{ \begin{aligned}
\frac{\partial^2 A}{\partial y^i \partial y^j} &= \frac{\partial}{\partial y^i} \left(\frac{1}{\alpha} \frac{\partial \Omega}{\partial y^j} - \frac{\Omega}{\alpha^2} \frac{y_j}{\alpha} \right) \langle [v, y]_m, y \rangle + \frac{\Omega}{\alpha} (\langle [v, v_j]_m, y \rangle + \langle [v, y]_m, v_j \rangle) \\
&= \left(\frac{1}{\alpha} \Omega_{ij} - \frac{y_i}{\alpha^3} \Omega_j - \frac{y_j}{\alpha^3} \Omega_i - \frac{\Omega}{\alpha^3} \delta_i^j + \frac{3\Omega}{\alpha^5} y_i y_j \right) \langle [v, y]_m, y \rangle + \left(\frac{1}{\alpha} \Omega_j - \frac{\Omega y_j}{\alpha^3} \right) \\
&\quad (\langle [v, v_i]_m, y \rangle + \langle [v, y]_m, v_i \rangle) + \left(\frac{1}{\alpha} \Omega_i - \frac{\Omega y_i}{\alpha^3} \right) \\
&\quad (\langle [v, v_j]_m, y \rangle + \langle [v, y]_m, v_j \rangle) + \frac{\Omega}{\alpha} (\langle [v, v_j]_m, v_i \rangle + \langle [v, v_i]_m, v_j \rangle).
\end{aligned} \right. \quad (7.2.5)$$

Now we wish to calculate $\frac{\partial^2 B}{\partial y^i \partial y^j}$.

Consider

$$\left\{ \begin{aligned} \frac{\partial B}{\partial y^j} &= \frac{\partial}{\partial y^j} \left\{ \Omega \left(\frac{e^u + e^{-u} - ue^{-u}}{e^u - ue^u + u^2 e^{-u}} \right) < [v, y]_m, v > \right\} \\ &= \left\{ \left(\frac{e^u + e^{-u} - ue^{-u}}{e^u - ue^u + u^2 e^{-u}} \right) \Omega_j + \Omega \left(\frac{e^{2u} + 2u - 2 - 2ue^{-2u} + u^2 e^{-2u}}{(e^u - ue^u + u^2 e^{-u})^2} \right) u_{yj} \right\} \\ &< [v, y]_m, v > + \left(\frac{e^u + e^{-u} - ue^{-u}}{e^u - ue^u + u^2 e^{-u}} \right) \Omega < [v, v_j]_m, v > . \end{aligned} \right.$$

$$\left\{ \begin{aligned} \frac{\partial^2 B}{\partial y^j \partial y^i} &= \left\{ \left(\frac{e^u + e^{-u} - ue^{-u}}{e^u - ue^u + u^2 e^{-u}} \right) \Omega_{ij} + \right. \\ &\left(\frac{e^{2u} + 2u - 2 - 2ue^{-2u} + u^2 e^{-2u}}{(e^u - ue^u + u^2 e^{-u})^2} \right) \Omega_j u_{yi} \\ &+ \Omega_i \left(\frac{e^{2u} + 2u - 2 - 2ue^{-2u} + u^2 e^{-2u}}{(e^u - ue^u + u^2 e^{-u})^2} \right) u_{yj} \\ &+ \Omega u_{yj} \left(\frac{e^{2u} + 2u - 2 - 2ue^{-2u} + u^2 e^{-2u}}{(e^u - ue^u + u^2 e^{-u})^2} \right) \\ &+ \Omega u_{yj} u_{yi} \frac{A_1}{A_2} \left. \right\} < [v, y]_m, v > + \left\{ \left(\frac{e^u + e^{-u} - ue^{-u}}{e^u - ue^u + u^2 e^{-u}} \right) \Omega_i \right. \\ &+ \left(\frac{e^{2u} + 2u - 2 - 2ue^{-2u} + u^2 e^{-2u}}{(e^u - ue^u + u^2 e^{-u})^2} \right) u_{yi} \Omega \left. \right\} < [v, v_j]_m, v > \\ &+ \left\{ \left(\frac{e^u + e^{-u} - ue^{-u}}{e^u - ue^u + u^2 e^{-u}} \right) \Omega_j + \Omega \left(\frac{e^{2u} + 2u - 2 - 2ue^{-2u} + u^2 e^{-2u}}{(e^u - ue^u + u^2 e^{-u})^2} \right) u_{yj} \right\} \\ &< [v, v_i]_m, v > . \end{aligned} \right. \quad (7.2.6)$$

Using equation (7.2.5) and equation (7.2.6), we get

$$\left\{ \begin{aligned} E_{i,j}(H, y) &= \frac{1}{2} \left[\frac{\partial}{\partial y^i} \left(\frac{1}{\alpha} \frac{\partial \Omega}{\partial y^j} - \frac{\Omega}{\alpha^2} \frac{y_j}{\alpha} \right) < [v, y]_m, y > + \frac{\Omega}{\alpha} \right. \\ &(< [v, v_j]_m, y > + < [v, y]_m, v_j >) = \left(\frac{1}{\alpha} \Omega_{ij} - \frac{y_i}{\alpha^3} \Omega_j - \frac{y_j}{\alpha^3} \Omega_i \right. \\ &- \frac{\Omega}{\alpha^3} \delta_i^j + \frac{3\Omega}{\alpha^5} y_i y_j) < [v, y]_m, y > + \left(\frac{1}{\alpha} \Omega_j - \frac{\Omega y_j}{\alpha^3} \right) \\ &(< [v, v_i]_m, y > + < [v, y]_m, v_i >) + \left(\frac{1}{\alpha} \Omega_i - \frac{\Omega y_i}{\alpha^3} \right) \\ &(< [v, v_j]_m, y > + < [v, y]_m, v_j >) + \frac{\Omega}{\alpha} (< [v, v_j]_m, v_i > \\ &+ < [v, v_i]_m, v_j >) + \left. \left\{ \left(\frac{e^u + e^{-u} - ue^{-u}}{e^u - ue^u + u^2 e^{-u}} \right) \Omega_{ij} \right\} \right. \end{aligned} \right. \quad (7.2.7)$$

$$\left\{ \begin{aligned}
& + \left(\frac{e^{2u} + 2u - 2 - 2ue^{-2u} + u^2 e^{-2u}}{(e^u - ue^u + u^2 e^{-u})^2} \right) \Omega_j u_{y^i} + \Omega_i \\
& \left(\frac{e^{2u} + 2u - 2 - 2ue^{-2u} + u^2 e^{-2u}}{(e^u - ue^u + u^2 e^{-u})^2} \right) u_{y^j} + \Omega u_{y^i y^j} \\
& \left(\frac{e^{2u} + 2u - 2 - 2ue^{-2u} + u^2 e^{-2u}}{(e^u - ue^u + u^2 e^{-u})^2} \right) + \Omega u_{y^j} u_{y^i} \frac{A_1}{A_2} \Bigg\} < [v, y]_m, v > \\
& + \left\{ \left(\frac{e^u + e^{-u} - ue^{-u}}{e^u - ue^u + u^2 e^{-u}} \right) \Omega_i + \left(\frac{e^{2u} + 2u - 2 - 2ue^{-2u} + u^2 e^{-2u}}{(e^u - ue^u + u^2 e^{-u})^2} \right) \right. \\
& \left. u_{y^i} \Omega \right\} < [v, v_j]_m, v > + \left\{ \left(\frac{e^u + e^{-u} - ue^{-u}}{e^u - ue^u + u^2 e^{-u}} \right) \Omega_j \right. \\
& \left. + \Omega \left(\frac{e^{2u} + 2u - 2 - 2ue^{-2u} + u^2 e^{-2u}}{(e^u - ue^u + u^2 e^{-u})^2} \right) u_{y^j} \right\} \\
& < [v, v_i]_{m, v} > \Bigg].
\end{aligned} \right.$$

□

CHAPTER 8

On $\mathbb{R}$ -Complex Finsler Space with Special Finsler Metrics

The study of $\mathbb{R}$ -complex Finsler spaces represents an important extension of classical Finsler geometry. In this chapter, we focus on $\mathbb{R}$ -complex Finsler spaces endowed with the cubic (α, β) -metric. We establish the fundamental metric structure by deriving the metric tensor fields $\tilde{g}_{ij}$ and $\tilde{g}_{i\bar{j}}$, computing their determinants and inverses, and studying geometric properties that characterize these novel spaces.

8.1 Introduction

The concept of complex Finsler space was introduced by Rizza [109] by generalizing the homogeneity condition of the fundamental function for complex scalars λ with respect to the fibre variables, and the first example of the complex hyperbolic geometry was introduced by Kobayashi, and that metric function is known as Kobayashi metric which gave an impulse to the study of complex Finsler geometry. Hermitian metrics on the base manifold, the locally Minkowski complex metrics, η -Einstein metrics etc are some interesting examples of complex Finsler metrics. Furthermore, Munteanu and Purcuru [90] extended the concept of complex Finsler space and introduced the concept of $\mathbb{R}$ -complex Finsler spaces by restricting the scalars to $\lambda \in \mathbb{R}$, i.e., to real scalars. Alda and Munteanu [7] studied $\mathbb{R}$ -complex Finsler spaces for the Finsler (α, β) -metric and determined the basic fundamental metric function and properties of Hermitian space with respect to the $\mathbb{R}$ -complex Finsler spaces with (α, β) -metric. Furthermore, some authors ([6], [104], [119], [120], [143], [147]) have also studied $\mathbb{R}$ -complex Finsler spaces with special metrics and obtained interesting geometric properties and various results on $\mathbb{R}$ -complex Finsler spaces.

8.2 Preliminaries

Let us consider a smooth complex manifold M of dimension n and let z^k be the local complex co-ordinates in a chart (U, ϕ) then the complexification of the real tangent bundle $T_c M$ splits into the sum of holomorphic tangent bundle $T' M$ of (1,0)-type vectors and its conjugate tangent bundle $T'' M$. The bundle $T' M$ is a complex manifold and the local coordinates of the chart $u \in T' M$ will be denoted by (z^k, η^k) . Now the change of coordinates in the respective local charts will be determined by

$$z'^k = z'^k(z), \quad \eta'^k = \frac{\partial z'^k}{\partial z^j} \eta^j. \quad (8.2.1)$$

Roughly speaking, the geometry of a complex Finsler space is the study of the geometric objects of a smooth complex manifold $T' M$ equipped with the Hermitian metric structure defined by $g_{i\bar{j}}$. Since the complexified tangent bundle of $T' M$ is decomposed as $T_c(T' M) = T'(T' M) \oplus T''(T' M)$, a natural local frame for $T'(T' M)$ is $\left\{ \frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k} \right\}$, and it changes according to the rules given below:

$$\frac{\partial}{\partial z^k} = \frac{\partial z'^k}{\partial z^h} \frac{\partial}{\partial z'^k} + \frac{\partial^2 z'^k}{\partial z^j \partial z^h} \eta^j \frac{\partial}{\partial \eta'^k}; \quad \frac{\partial}{\partial \eta^k} = \frac{\partial z'^k}{\partial z^h} \frac{\partial}{\partial \eta'^k}.$$

From the above, it is clear that the transformation rule of $\frac{\partial}{\partial z^k}$ involves second-order partial derivatives.

A complex nonlinear connection, briefly (c.n.c.), is a supplementary distribution $H(T' M)$ complementary to the vertical distribution $V(T' M)$ in $T'(T' M)$. The vertical distribution is spanned by $\frac{\partial}{\partial \eta^k}$, and an adapted frame in $H(T' M)$ is given by $\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}$, where N_k^j are the coefficients of the complex nonlinear connection and they satisfy a certain transformation rule at (8.2.1), so that $\frac{\delta}{\delta z^k}$ transforms like a vector on the base manifold M . Next, we use the abbreviations $\partial_k = \frac{\partial}{\partial z^k}$, $\delta_k = \frac{\delta}{\delta z^k}$, $\dot{\partial}_k = \frac{\partial}{\partial \eta^k}$ and $\partial_{\bar{k}}$, $\dot{\partial}_{\bar{k}}$, $\delta_{\bar{k}}$ for their conjugates. The dual adapted basis of δ_k , $\dot{\partial}_k$ are $\{dz^k, \delta \eta^k = d\eta^k + N_j^k dz^j\}$ and $\{d\bar{z}^k, \delta \bar{\eta}^k\}$ for their conjugates.

We recall that the homogeneity of the metric function of a complex Finsler space [89, 147] is with respect to complex scalars and the metric tensor of the space is a Hermitian one. In [90], the definition of complex Finsler spaces was slightly modified as

Definition 8.2.1. An $\mathbb{R}$ -complex Finsler metric on M is a continuous function $F : T'M \rightarrow \mathbb{R}$ that satisfying the following conditions:

1. The metric $L = F^2$ is positively 2-homogeneous function on $T'M/0$;
2. $F(z, \eta) \geq 0$, with equality holding if and only if $\eta = 0$;
3. $F(z, \lambda \eta, \bar{z}, \lambda \bar{\eta}) = |\lambda| F(z, \eta, \bar{z}, \bar{\eta})$, for all $\lambda \in \mathbb{R}$;
4. the Hermitian matrix $g_{i\bar{j}}(z, \eta)$ with $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ is positive definite on $T'M$ and it is called the fundamental metric tensor.

It follows that L is $(2, 0)$ -homogeneous with respect to the real scalar λ , and it is proved that the following identities are fulfilled [104]:

$$\frac{\partial L}{\partial \eta^i} \eta^i + \frac{\partial L}{\partial \bar{\eta}^i} \bar{\eta}^i = 2L; \quad g_{ij} \eta^i + g_{\bar{j}i} \bar{\eta}^i = \frac{\partial L}{\partial \eta^j}, \quad (8.2.2)$$

$$\frac{\partial g_{ik}}{\partial \eta^j} \eta^j + \frac{\partial g_{ij}}{\partial \bar{\eta}^j} \bar{\eta}^j = 0; \quad \frac{\partial g_{i\bar{k}}}{\partial \eta^j} \eta^j + \frac{\partial g_{i\bar{k}}}{\partial \bar{\eta}^j} \bar{\eta}^j = 0, \quad (8.2.3)$$

$$2L = g_{ij} \eta^i \eta^j + g_{\bar{i}\bar{j}} \bar{\eta}^i \bar{\eta}^j + 2g_{i\bar{j}} \eta^i \bar{\eta}^j, \quad (8.2.4)$$

where,

$$g_{ij} = \frac{\partial^2 L}{\partial \eta^i \partial \eta^j}; \quad g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}; \quad g_{\bar{i}\bar{j}} = \frac{\partial^2 L}{\partial \bar{\eta}^i \partial \bar{\eta}^j}.$$

Definition 8.2.2. An $\mathbb{R}$ -complex Finsler space (M, F) is called an $\mathbb{R}$ -complex Finsler space with (α, β) -metric if the fundamental function $F(z, \eta, \bar{z}, \bar{\eta})$ is $\mathbb{R}$ -homogeneous in terms of the functions $\alpha(z, \eta, \bar{z}, \bar{\eta})$ and $\beta(z, \eta, \bar{z}, \bar{\eta})$, which depend on $z^i, \eta^i, \bar{z}^i$ and $\bar{\eta}^i$, for $i = 1, 2, \dots, n$. That is,

$$F(z, \eta, \bar{z}, \bar{\eta}) = F(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})), \quad (8.2.5)$$

where,

$$\alpha^2(z, \eta, \bar{z}, \bar{\eta}) = \frac{1}{2}(a_{ij}\eta^i\eta^j + a_{i\bar{j}}\bar{\eta}^i\bar{\eta}^j + 2a_{i\bar{j}}\eta^i\bar{\eta}^j) = \text{Re}\{a_{ij}\eta^i\eta^j + a_{i\bar{j}}\eta^i\bar{\eta}^j\}, \quad (8.2.6)$$

$$\beta(z, \eta, \bar{z}, \bar{\eta}) = \frac{1}{2}(b_i\eta^i + b_{\bar{i}}\bar{\eta}^{\bar{i}}) = \text{Re}(b_i\eta^i),$$

with $a_{ij} = a_{ij}(z)$, $a_{i\bar{j}} = a_{i\bar{j}}(z)$, $b_i = b_i(z)$. We denote

$$L(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})) = F^2(\alpha(z, \eta, \bar{z}, \bar{\eta}), \beta(z, \eta, \bar{z}, \bar{\eta})). \quad (8.2.7)$$

Definition 8.2.3. An $\mathbb{R}$ -complex Finsler space (M, F) is called a Hermitian space, if the tensor $g_{ij} = 0$ and the Hermitian metric $g_{i\bar{j}}$ is invertible. An $\mathbb{R}$ -Complex Finsler space (M, F) is called a non-Hermitian space if the metric tensor $g_{i\bar{j}} = 0$ and the Hermitian metric g_{ij} is invertible. Here, g_{ij} and $g_{i\bar{j}}$ are the metric tensors of the space and defined respectively by, $g_{ij} = \frac{\partial}{\partial \eta^i} \frac{\partial}{\partial \eta^j} L$ and $g_{i\bar{j}} = \frac{\partial}{\partial \eta^i} \frac{\partial}{\partial \bar{\eta}^j} L$.

8.3 $\mathbb{R}$ -Complex Cubic (α, β) -Metric

The $\mathbb{R}$ -complex Finsler space produces two types of fundamental metric tensor fields one is g_{ij} other is $g_{i\bar{j}}$ and in Hermitian geometry the tensor field $g_{i\bar{j}}$ must be invertible. A space is said to be Hermitian space if $\det(g_{i\bar{j}}) \neq 0$ in $\mathbb{R}$ -complex Finsler spaces and it is said to be non-Hermitian space if $\det(g_{ij}) \neq 0$ in $\mathbb{R}$ -Complex Finsler spaces. In the present section, we determine the fundamental metric tensor for $\mathbb{R}$ -Complex Finsler space equipped with a cubic (α, β) -metric and obtained condition for the property of non-Hermitian $\mathbb{R}$ -Complex Finsler spaces. Now consider an $\mathbb{R}$ -Complex Finsler space with the metric

$$L(\alpha, \beta) = \frac{(\alpha + \beta)^6}{\alpha^4}. \quad (8.3.1)$$

Then, it follows that $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ which is the cubic (α, β) -metric. Now, we compute the following quantities for $\mathbb{R}$ -Complex Finsler spaces with the cubic (α, β) -metric $F = \frac{(\alpha + \beta)^3}{\alpha^2}$.

$$\alpha L_\alpha + \beta L_\beta = 2L, \quad \alpha L_{\alpha\alpha} + \beta L_{\alpha\beta} = L_\alpha, \quad (8.3.2)$$

$$\alpha L_{\alpha\beta} + \beta L_{\beta\beta} = L_\beta, \quad \alpha^2 L_{\alpha\alpha} + 2\alpha\beta L_{\alpha\beta} + \beta^2 L_{\beta\beta} = 2L, \quad (8.3.3)$$

where,

$$L_\alpha = \frac{\partial L}{\partial \alpha}, \quad L_\beta = \frac{\partial L}{\partial \beta}, \quad L_{\alpha\beta} = \frac{\partial^2 L}{\partial \alpha \partial \beta}, \quad L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha^2}, \quad L_{\beta\beta} = \frac{\partial^2 L}{\partial \beta^2}. \quad (8.3.4)$$

$$L_\alpha = \frac{6\alpha(\alpha + \beta)^5 - 4(\alpha + \beta)^6}{\alpha^5}, \quad (8.3.5)$$

$$L_\beta = \frac{6(\alpha + \beta)^5}{\alpha^4}, \quad (8.3.6)$$

$$L_{\alpha\alpha} = \frac{-48\alpha(\alpha + \beta)^5 + 30\alpha^2(\alpha + \beta)^4 + 20(\alpha + \beta)^6}{\alpha^6}, \quad (8.3.7)$$

$$L_{\beta\beta} = \frac{30(\alpha + \beta)^4}{\alpha^4}, \quad (8.3.8)$$

$$L_{\alpha\beta} = \frac{30\alpha(\alpha + \beta)^4 - 24(\alpha + \beta)^5}{\alpha^5}, \quad (8.3.9)$$

$$L_{\beta\alpha} = \frac{30\alpha(\alpha + \beta)^4 - 24(\alpha + \beta)^5}{\alpha^5}, \quad (8.3.10)$$

$$\left\{ \begin{aligned} \alpha L_\alpha + \beta L_\beta &= \alpha \left(\frac{6\alpha(\alpha + \beta)^5 - 4(\alpha + \beta)^6}{\alpha^5} \right) + \frac{6\beta(\alpha + \beta)^5}{\alpha^4}, \\ &= \frac{(\alpha + \beta)^5(2\alpha + 2\beta)}{\alpha^4} = \frac{2(\alpha + \beta)^6}{\alpha^4} = 2L, \end{aligned} \right. \quad (8.3.11)$$

$$\left\{ \begin{aligned} \alpha L_{\alpha\beta} + \beta L_{\beta\beta} &= \alpha \left(\frac{30\alpha(\alpha + \beta)^4 - 24(\alpha + \beta)^5}{\alpha^5} \right) + \frac{30\beta(\alpha + \beta)^4}{\alpha^4}, \\ &= \frac{(\alpha + \beta)^4(6\alpha + 6\beta)}{\alpha^4} = \frac{6(\alpha + \beta)^5}{\alpha^4} = L_\beta. \end{aligned} \right. \quad (8.3.12)$$

$$\left\{ \begin{aligned} \alpha L_{\alpha\alpha} + \beta L_{\alpha\beta} &= \frac{1}{\alpha^5} [(-48\alpha(\alpha + \beta)^5 + 30\alpha^2(\alpha + \beta)^4 + 20(\alpha + \beta)^6) \\ &\quad + \beta(30\alpha(\alpha + \beta)^4 - 24(\alpha + \beta)^5)] = \frac{(\alpha + \beta)^4}{\alpha^5} \\ [2\alpha^2 - 2\alpha\beta - 4\beta^2] &= L_\alpha. \end{aligned} \right. \quad (8.3.13)$$

$$\left\{ \begin{aligned} \alpha^2 L_{\alpha\alpha} + 2\alpha\beta L_{\alpha\beta} + \beta^2 L_{\beta\beta} &= \frac{1}{\alpha^4} (-48\alpha(\alpha + \beta)^5 + 30\alpha^2(\alpha + \beta)^4 + 20(\alpha + \beta)^6) \\ &\quad + \frac{2\beta}{\alpha^4} (30\alpha(\alpha + \beta)^4 - 24(\alpha + \beta)^5) + \frac{30\beta^2(\alpha + \beta)^4}{\alpha^4} \\ &= \frac{2(\alpha + \beta)^6}{\alpha^4} = 2L. \end{aligned} \right. \quad (8.3.14)$$

We propose to determine the metric tensors of an $\mathbb{R}$ -complex Finsler space using the following identities:

$$g_{ij} = \frac{\partial^2 L(z, \eta, \bar{z}, \lambda \bar{\eta})}{\partial \eta^i \partial \eta^j}, \quad g_{i\bar{j}} = \frac{\partial^2 L(z, \eta, \bar{z}, \lambda \bar{\eta})}{\partial \eta^i \partial \bar{\eta}^j}.$$

Each of these will be of interest in the subsequent sections.

We consider,

$$\frac{\partial \alpha}{\partial \eta^i} = \frac{1}{2\alpha} (a_{ij} \eta^j + a_{i\bar{j}} \bar{\eta}^j) = \frac{1}{2\alpha l_i}, \quad \frac{\partial \beta}{\partial \eta^i} = \frac{1}{2} b_i,$$

$$\frac{\partial \alpha}{\partial \bar{\eta}^i} = \frac{1}{2\alpha} (a_{i\bar{j}} \bar{\eta}^j + a_{ij} \eta^j) = \frac{\partial \beta}{\partial \bar{\eta}^i} = \frac{1}{2} b_{\bar{i}},$$

where,

$l_i = (a_{ij}\eta^j + a_{i\bar{j}}\bar{\eta}^{\bar{j}}), l_{\bar{j}} = a_{i\bar{j}}\bar{\eta}^{\bar{i}} + a_{ij}\eta^i$. We find immediately, $l_i\eta^i + l_{\bar{j}}\bar{\eta}^j = 2\alpha^2$.

We denote:

$$\eta^i = \frac{\partial L}{\partial \eta^i} = \frac{\partial}{\partial \eta^i} F^2 = 2F \frac{\partial}{\partial \eta^i} \left(\frac{(\alpha + \beta)^3}{\alpha^2} \right),$$

$$\eta_i = \rho_0 l_i + \rho_1 b_i,$$

where,

$$\rho_0 = \frac{1}{2} \alpha^{-1} L_\alpha, \quad (8.3.15)$$

and

$$\rho_1 = \frac{1}{2} L_\beta. \quad (8.3.16)$$

Differentiating ρ_0 and ρ_1 with respect to η^j and $\bar{\eta}^j$, respectively, we obtain:

$$\frac{\partial \rho_0}{\partial \eta^j} = \rho_{-2} l_j + \rho_{-1} b_j,$$

and

$$\frac{\partial \rho_0}{\partial \bar{\eta}^j} = \rho_{-2} l_{\bar{j}} + \rho_{-1} b_{\bar{j}}.$$

Similarly $\frac{\partial \rho_1}{\partial \eta^i} = \eta_{-1} l_i + \mu_0 b_i$, $\frac{\partial \rho_1}{\partial \bar{\eta}^i} = \rho_{-1} l_{\bar{i}} + \mu_0 b_{\bar{i}}$,

where,

$$\rho_{-2} = \frac{\alpha L_{\alpha\alpha} - L_\alpha}{4\alpha^3}, \quad \rho_{-1} = \frac{L_{\alpha\beta}}{4\alpha}, \quad \mu_0 = \frac{L_{\beta\beta}}{4}. \quad (8.3.17)$$

By direct computation, using (8.3.15), (8.3.16), and (8.3.17), we obtain the following result.

Theorem 8.3.1. *The invariants of an $\mathbb{R}$ -complex Finsler space with cubic (α, β) -metric, namely ρ_0 , ρ_1 , ρ_{-2} , ρ_{-1} , and μ_0 , are given by:*

$$\left\{ \begin{array}{l} \rho_0 = \frac{1}{2}\alpha^{-1}L\alpha = \frac{1}{2\alpha^6} (6\alpha(\alpha + \beta)^5 - 4(\alpha + \beta)^6), \\ \rho_1 = \frac{1}{2}L\beta = \frac{3(\alpha + \beta)^5}{\alpha^4}, \\ \rho_{-2} = \frac{\alpha L_{\alpha\alpha} - L_{\alpha}}{4\alpha^3} = \frac{-27\alpha(\alpha + \beta)^5 + 15\alpha^2(\alpha + \beta)^4 + 12(\alpha + \beta)^6}{2\alpha^8}, \\ \rho_{-1} = \frac{L_{\alpha\beta}}{4\alpha} = \frac{30\alpha(\alpha + \beta)^4 - 24(\alpha + \beta)^5}{4\alpha^6}, \\ \mu_0 = \frac{L_{\beta\beta}}{4} = \frac{30(\alpha + \beta)^4}{4\alpha^4}. \end{array} \right.$$

Subscripts 0, 1, -2, -1 and 0 gives us the degree of homogeneity of these invariants.

8.4 Fundamental Tensor of $\mathbb{R}$ -Complex Finsler Space with Cubic (α, β) -Metric

The fundamental metric tensors of an $\mathbb{R}$ -complex Finsler space with cubic (α, β) -metric are given by

$$g_{ij} = \rho_0 a_{ij} + \rho_{-2} l_i l_j + \mu_0 b_i b_j + \rho_{-1} (b_j l_i + b_i l_j). \quad (8.4.1)$$

From Theorem (8.3.1), we have

$$\left\{ \begin{array}{l} g_{ij} = \frac{1}{2\alpha^6} (6\alpha(\alpha + \beta)^5 - 4(\alpha + \beta)^6) a_{ij} + \\ \left(\frac{-27\alpha(\alpha + \beta)^5 + 15\alpha^2(\alpha + \beta)^4 + 12(\alpha + \beta)^6}{2\alpha^8} \right) l_i l_j + \frac{30(\alpha + \beta)^4}{4\alpha^4} b_i b_j \\ + \frac{30\alpha(\alpha + \beta)^4 - 24(\alpha + \beta)^5}{4\alpha^6} (b_j l_i + b_i l_j). \end{array} \right. \quad (8.4.2)$$

$$\left\{ \begin{aligned} \tilde{g}_{i\bar{j}} &= \frac{1}{2\alpha^6} (6\alpha(\alpha + \beta)^5 - 4(\alpha + \beta)^6) a_{i\bar{j}} \\ &+ \left(\frac{-27\alpha(\alpha + \beta)^5 + 15\alpha^2(\alpha + \beta)^4 + 12(\alpha + \beta)^6}{2\alpha^8} \right) l_i l_{\bar{j}} \\ &+ \frac{30(\alpha + \beta)^4}{4\alpha^4} b_i b_{\bar{j}} + \frac{30\alpha(\alpha + \beta)^4 - 24(\alpha + \beta)^5}{4\alpha^6} (b_{\bar{j}} l_i + b_i l_{\bar{j}}). \end{aligned} \right. \quad (8.4.3)$$

Or, equivalently,

$$\tilde{g}_{ij} = \rho_0 [a_{ij} + p l_i l_j + q b_i b_j + r \eta_i \eta_j], \quad (8.4.4)$$

$$\tilde{g}_{i\bar{j}} = \rho_0 [a_{i\bar{j}} + p l_i l_{\bar{j}} + q b_i b_{\bar{j}} + r \eta_i \eta_{\bar{j}}], \quad (8.4.5)$$

where, $\rho_0 = \frac{1}{2} \alpha^{-1} L_\alpha$.

$$p = \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)}, \quad (8.4.6)$$

$$q = \frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2} \quad (8.4.7)$$

$$r = \frac{3\alpha - 12\beta}{2\alpha^2 - 2\alpha\beta - 4\beta^2}. \quad (8.4.8)$$

The next objective is to obtain the determinant and the inverse of the tensor field $\widetilde{g}_{ij}$. The solution of the non-singular non-Hermitian metric $\widetilde{Q}_{ij}$ is given as follows. The following proposition is proved in [80].

Proposition 8.4.1. Suppose:

1. (Q_{ij}) is a non-singular $n \times n$ complex matrix with inverse Q^{ji} ;
2. C_i and $C_{\bar{i}} = \tilde{C}_i$, $i = 1, \dots, n$ are complex numbers;
3. $C^i := Q^{ji} C_j$ and its conjugates: $C^2 := C^i C_i = \tilde{C}^i C_{\bar{i}}$; $H_{ij} := Q_{ij} \pm C_i C_j$.

Then,

$$1. \det(H_{ij}) = (1 \pm C^2) \det(Q_{ij}),$$

2. Whenever $(1 \pm C^2) \neq 0$, the metrix (H_{ij}) is invertible and in this case its inverse is

$$H^{ji} = Q^{ji} \pm \frac{1}{1 \pm C^2} C^i C^j.$$

Using the above proposition, we prove the following theorem:

Theorem 8.4.2. For a non-Hermitian $\mathbb{R}$ -complex Finsler space with cubic (α, β) -metric $F = \frac{(\alpha + \beta)^3}{\alpha^2}$, the following identities hold:

1. The contravariant tensor $\tilde{g}^{ij}$ of the fundamental tensor $\tilde{g}_{ij}$ is:

$$\tilde{g}^{ji} = \frac{2\alpha^6}{(6\alpha(\alpha + \beta)^5 - 4(\alpha + \beta)^6)} a^{ji} + A_1 \eta^i \eta^j + A_2 (b^i \eta^j + b^j \eta^i) + A_3 b^i b^j + \frac{A_4}{A_5},$$

where,

$$A_1 = \frac{h_1}{h_2} + \frac{h_3}{h_4},$$

$$h_1 = -3\alpha\beta + 12\beta^2, \quad h_2 = 2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma,$$

$$h_3 = 15\alpha^2\epsilon^2(-3\alpha\beta + 12\beta^2)^2,$$

$$h_4 = (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) \times [(2\alpha^2 - 2\alpha\beta - 4\beta^2) \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \epsilon^2(-3\alpha\beta + 12\beta^2))\}],$$

$$A_2 = \frac{h_5}{h_6},$$

$$h_5 = 15\alpha^2(-3\alpha\beta + 12\beta^2)\epsilon,$$

$$h_6 = (2\alpha^2 - 2\alpha\beta - 4\beta^2) \times [(2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \varepsilon^2(-3\alpha\beta + 12\beta^2))\}],$$

$$A_3 = \frac{h_7}{h_8},$$

$$h_7 = 15\alpha^2,$$

$$h_8 = 2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2(\omega + \frac{\varepsilon^2(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma}),$$

$$A_4 = (1 + \gamma k_1 + \varepsilon k_2)^2 \eta^i \eta^j + (1 + \gamma k_1 + \varepsilon k_2) \times (\frac{r_7}{r_8} + \frac{r_9}{r_{10}})(\eta^i b^j + \eta^j b^i) + (\frac{r_7}{r_8} + \frac{r_9}{r_{10}})^2 b^i b^j,$$

$$A_5 = 1 + (\gamma(1 + \gamma k_1 + \varepsilon k_2) + \varepsilon(\frac{r_7}{r_8} + \frac{r_9}{r_{10}}))\sqrt{\frac{3\alpha - 12\beta}{2\alpha^2 - 2\alpha\beta - 4\beta^2}},$$

$$k_1 = \frac{r_1}{r_2} + \frac{r_3}{r_4},$$

$$r_1 = -3\alpha\beta + 12\beta^2, \quad r_2 = 2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma,$$

$$r_3 = 15\alpha^2 \varepsilon^2 (-3\alpha\beta + 12\beta^2)^2,$$

$$r_4 = (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) \times [(2\alpha^2 - 2\alpha\beta - 4\beta^2) \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \varepsilon^2(-3\alpha\beta + 12\beta^2))\}],$$

$$k_2 = \frac{r_5}{r_6},$$

$$r_5 = (-45\alpha\beta + 180\beta^2) \times \alpha^6 \times (2\alpha^2 - 2\alpha\beta - 4\beta^2)^2,$$

$$r_6 = [(2\alpha^2 - 2\alpha\beta - 4\beta^2) \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \varepsilon^2(-3\alpha\beta + 12\beta^2)) \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma)^2,$$

$$r_7 = 15\varepsilon\alpha^2,$$

$$r_8 = 2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2(\omega + \frac{\varepsilon^2(-3\alpha\beta + 12\beta^2)}{(2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma)}),$$

$$r_9 = \varepsilon\gamma(-45\alpha\beta + 180\beta^2)\alpha^2,$$

$$r_{10} = \{2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2(\omega + \frac{\varepsilon^2(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma})\} \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma).$$

$$\begin{aligned} 2. \det(a_{ij} + \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)}l_{ilj} + \frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2}b_ib_j + \frac{3\alpha - 12\beta}{2\alpha^2 - 2\alpha\beta - 4\beta^2}\eta_i\eta_j) \\ = [1 + (\gamma(1 + \gamma k_1 + \varepsilon k_2) + \varepsilon(\frac{r_7}{r_8} + \frac{r_9}{r_{10}}))\sqrt{\frac{3\alpha - 12\beta}{2\alpha^2 - 2\alpha\beta - 4\beta^2}}] \\ \times [1 + \omega + \frac{(-3\alpha\beta + 12\beta^2)\varepsilon^2}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma}] \times (1 + 2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma)\det(a_{ij}). \end{aligned}$$

Proof. Step 1: We prove this theorem in the following three steps: We write $\widetilde{g}_{ij}$ from (8.4.4) in the following form:

$$\widetilde{g}_{ij} = \rho_0[a_{ij} + pl_i l_j + ql_i l_j + r\eta_i \eta_j]. \quad (8.4.9)$$

We take $\tilde{Q}_{ij} = a_{ij}$ and $\tilde{C}_i = \sqrt{\frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)}} l_i$. By applying the Proposition

(8.4.1) we obtain $\tilde{Q}^{ij} = a^{ij}$, $\tilde{C}^2 = \tilde{C}^i \tilde{C}_i = \sqrt{\frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)}} l_i \times \tilde{Q}^{ji} \times \tilde{C}_j = \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} \times l_i l_j a^{ji} = \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} \gamma$, and $1 + \tilde{C}^2 = (1 + \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} \gamma)$. So, the matrix $\tilde{H}_{ij} = a_{ij} - \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} l_i l_j$, is invertible with

$$\tilde{H}^{ij} = a^{ji} + \frac{(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma} \eta^i \eta^j$$

$$\det(a_{ij} + \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} l_i l_j) = (1 + \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} \gamma) \det(a_{ij}).$$

Step 2: Now, we consider $\tilde{Q}_{ij} = a_{ij} + \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} l_i l_j$, and $\tilde{C}_i = \sqrt{\frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2}} b_i$,
By applying the Proposition (8.4.1) we have,

$$\tilde{Q}^{ji} = a^{ji} + \frac{(-3\alpha\beta + 12\beta^2) \eta^i \eta^j}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma},$$

$$\tilde{C}^2 = \tilde{C}^i \tilde{C}_i = \tilde{Q}^{ji} \times \tilde{C}_j \times \tilde{C}_i = \sqrt{\frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2}} b_i \left[a^{ij} + \frac{(-3\alpha\beta + 12\beta^2) \eta^i \eta^j}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma} \times \sqrt{\frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2}} b^j \right].$$

$$\tilde{C}^2 = \frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2} \left[\omega + \frac{(-3\alpha\beta + 12\beta^2) \varepsilon^2}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma} \right].$$

Therefore,

$$1 + \tilde{C}^2 \neq 0.$$

Where, $\varepsilon = b_j \eta^j$, $\omega = b_j b^j$. It results that the inverse $\tilde{H}_{ij} = a_{ij} + \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} l_i l_j + \frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2} b_i b_j$ exists and it is

$$\tilde{H}^{ji} = Q^{ji} + \frac{1}{1 + \tilde{C}^2} C^i C^j,$$

$$\tilde{H}^{ji} = a^{ji} + \frac{p \eta^i \eta^j}{1 + p\gamma} + \frac{q \left[b^i + \frac{p \varepsilon \eta^i}{1 + p\gamma} \right] \left[b^j + \frac{p \varepsilon \eta^j}{1 + p\gamma} \right]}{\tau} \quad (8.4.10)$$

$$\tilde{H}^{ji} = a^{ji} + \left(\frac{p}{1 + p\gamma} + \frac{qp^2 \varepsilon^2}{\tau(1 + p\gamma)^2} \right) \eta^i \eta^j + \frac{pq \varepsilon}{(1 + p\gamma) \tau} (b^i \eta^j + b^j \eta^i) + \frac{q}{\tau} b^i b^j. \quad (8.4.11)$$

where,

$$\tau = 1 + q \left[\omega + \frac{p\varepsilon^2}{1 + p\gamma} \right].$$

Now, putting value of p, q, τ and we get following equation

$$\tilde{H}^{ji} = a^{ji} + A_1 \eta^i \eta^j + A_2 (b^i \eta^j + b^j \eta^i) + A_3 b^i b^j, \quad (8.4.12)$$

where,

$$A_1 = \frac{h_1}{h_2} + \frac{h_3}{h_4},$$

$$h_1 = -3\alpha\beta + 12\beta^2, \quad h_2 = 2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma,$$

$$h_3 = 15\alpha^2\varepsilon^2(-3\alpha\beta + 12\beta^2)^2,$$

$$h_4 = (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) \times [(2\alpha^2 - 2\alpha\beta - 4\beta^2) \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \varepsilon^2(-3\alpha\beta + 12\beta^2))\}],$$

$$A_2 = \frac{h_5}{h_6},$$

$$h_5 = 15\alpha^2(-3\alpha\beta + 12\beta^2)\varepsilon,$$

$$h_6 = (2\alpha^2 - 2\alpha\beta - 4\beta^2) \times [(2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \varepsilon^2(-3\alpha\beta + 12\beta^2))\}],$$

$$A_3 = \frac{h_7}{h_8},$$

$$h_7 = 15\alpha^2,$$

$$h_8 = 2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2\left(\omega + \frac{\varepsilon^2(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma}\right).$$

$$\begin{aligned} \det[a_{ij} + \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)}l_il_j + \frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2}b_ib_j] &= \left[1 + \right. \\ &\frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2} \left(\omega + \frac{(-3\alpha\beta + 12\beta^2)\varepsilon^2}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma} \right) \left. \right] \\ &(1 + \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)}\gamma)\det(a_{ij}). \end{aligned} \quad (8.4.13)$$

Step 3: We put

$$\tilde{Q}^{ji} = a^{ji} + \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)}l_il_j + \frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2}b_ib_j \quad (8.4.14)$$

and $\tilde{C}_i = \sqrt{\frac{3\alpha - 12\beta}{2\alpha^2 - 2\alpha\beta - 4\beta^2}}\eta_i$, clearly observe that and obtain

$$\tilde{Q}^{ji} = a^{ji} + \left(\frac{p}{1+p\gamma} + \frac{qp^2\varepsilon^2}{\tau(1+p\gamma)^2} \right) \eta^i\eta^j + \frac{pq\varepsilon}{\tau(1+p\gamma)}(b^i\eta^j + b^j\eta^i) + \frac{q}{\tau}b^ib^j \quad (8.4.15)$$

and $\tilde{C}_i = M\eta^i + Nb^j$, where

$$M = \left[1 + \left(\frac{p}{1+p\gamma} + \frac{p^2 q \varepsilon^2}{\tau(1+p\gamma)^2} \right) \right] \gamma + \frac{pq\varepsilon}{\tau(1+p\gamma)^3}, \quad (8.4.16)$$

$$N = \frac{q\varepsilon}{\tau} + \frac{pq\varepsilon\gamma}{\tau(1+p\gamma)}. \quad (8.4.17)$$

Now, substituting the values of p , q , and τ , we get

$$M = 1 + \gamma k_1 + \varepsilon k_2,$$

where,

$$k_1 = \frac{r_1}{r_2} + \frac{r_3}{r_4},$$

$$r_1 = -3\alpha\beta + 12\beta^2, \quad r_2 = 2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma,$$

$$r_3 = 15\alpha^2\varepsilon^2(-3\alpha\beta + 12\beta^2)^2,$$

$$r_4 = (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) \times [(2\alpha^2 - 2\alpha\beta - 4\beta^2) \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \varepsilon^2(-3\alpha\beta + 12\beta^2))\}],$$

$$k_2 = \frac{r_5}{r_6}$$

$$r_5 = (-45\alpha\beta + 180\beta^2) \times \alpha^6 \times (2\alpha^2 - 2\alpha\beta - 4\beta^2)^2,$$

$$r_6 = [(2\alpha^2 - 2\alpha\beta - 4\beta^2) \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \varepsilon^2(-3\alpha\beta + 12\beta^2))\} \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma)^2.$$

Now, substituting the values of p , q , and τ , we get

$$N = \frac{r_7}{r_8} + \frac{r_9}{r_{10}},$$

where,

$$r_7 = 15\varepsilon\alpha^2, \quad r_8 = 2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2(\omega + \frac{\varepsilon^2(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma}),$$

$$r_9 = \varepsilon\gamma(-45\alpha\beta + 180\beta^2)\alpha^2,$$

$$r_{10} = \{2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2(\omega + \frac{\varepsilon^2(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma})\} \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma).$$

And

$$\tilde{C}^2 = (M\gamma + N\varepsilon)\sqrt{r},$$

Putting value of M and N , we get

$$\tilde{C}^2 = (\gamma(1 + \gamma k_1 + \varepsilon k_2) + \varepsilon(\frac{r_7}{r_8} + \frac{r_9}{r_{10}}))\sqrt{\frac{3\alpha - 12\beta}{2\alpha^2 - 2\alpha\beta - 4\beta^2}},$$

where,

$$k_1 = \frac{r_1}{r_2} + \frac{r_3}{r_4},$$

$$r_1 = -3\alpha\beta + 12\beta^2, \quad r_2 = 2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma,$$

$$r_3 = 15\alpha^2\varepsilon^2(-3\alpha\beta + 12\beta^2)^2,$$

$$r_4 = (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) \times [(2\alpha^2 - 2\alpha\beta - 4\beta^2) \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \varepsilon^2(-3\alpha\beta + 12\beta^2))\}],$$

$$k_2 = \frac{r_5}{r_6},$$

$$r_5 = (-45\alpha\beta + 180\beta^2) \times \alpha^6 \times (2\alpha^2 - 2\alpha\beta - 4\beta^2)^2,$$

$$r_6 = [(2\alpha^2 - 2\alpha\beta - 4\beta^2) \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \varepsilon^2(-3\alpha\beta + 12\beta^2))\} \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma)^2,$$

$$r_7 = 15\varepsilon\alpha^2,$$

$$r_8 = 2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2(\omega + \frac{\varepsilon^2(-3\alpha\beta + 12\beta^2)}{(2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma)}),$$

$$r_9 = \varepsilon\gamma(-45\alpha\beta + 180\beta^2)\alpha^2,$$

$$r_{10} = \{2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2(\omega + \frac{\varepsilon^2(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma})\} \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma).$$

$$1 + \tilde{C}^2 \neq 0,$$

Clearly, the matrix $\tilde{H}_{ij}$ is invertible.

$$\tilde{C}^i = a^{ji} + \left\{ \frac{p\eta^i\eta^j}{1+p\gamma} + \frac{q \left[b^i + \frac{p\varepsilon\eta^i}{1+p\gamma} \right] \left[b^j + \frac{p\varepsilon\eta^j}{1+p\gamma} \right]}{\tau} \right\} \eta_j,$$

and

$$\tilde{C}^j = a^{ji} + \left\{ \frac{p\eta^i\eta^j}{1+p\gamma} + \frac{q \left[b^i + \frac{p\varepsilon\eta^i}{1+p\gamma} \right] \left[b^j + \frac{p\varepsilon\eta^j}{1+p\gamma} \right]}{\tau} \right\} \eta_i,$$

where $\widetilde{\tilde{C}^i \tilde{C}^j} = M^2 \eta^i \eta^j + MN(\eta^i b^j + \eta^j b^i) + N^2 b^i b^j$. Again, by applying Proposition (8.4.1),

we obtain the inverse of $\widetilde{\tilde{H}_{ij}}$ as follows:

$$\tilde{H}^{ji} = a^{ji} + \left(\frac{p}{1+p\gamma} + \frac{qp^2\varepsilon^2}{\tau(1+p\gamma)^2} \right) \eta^i \eta^j + \frac{pq\varepsilon}{(1+p\gamma)\tau} (b^i \eta^j + b^j \eta^i) + \frac{q}{\tau} b^i b^j \quad (8.4.18)$$

$$+ \frac{M^2 \eta^i \eta^j + MN(\eta^i b^j + \eta^j b^i) + N^2 b^i b^j}{1 + (M\gamma + N\varepsilon)\sqrt{r}}.$$

Putting all value in equation (8.4.18) and we get,

$$\tilde{H}^{ij} = a^{ji} + A_1 \eta^i \eta^j + A_2 (b^i \eta^j + b^j \eta^i) + A_3 b^i b^j + \frac{A_4}{A_5}. \quad (8.4.19)$$

Where,

$$\begin{aligned} A_1 &= \frac{h_1}{h_2} + \frac{h_3}{h_4}, \\ h_1 &= -3\alpha\beta + 12\beta^2, \\ h_2 &= 2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma, \\ h_3 &= 15\alpha^2\varepsilon^2(-3\alpha\beta + 12\beta^2)^2, \\ h_4 &= (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) \times [(2\alpha^2 - 2\alpha\beta - 4\beta^2) \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \varepsilon^2(-3\alpha\beta + 12\beta^2))\}], \\ A_2 &= \frac{h_5}{h_6}, \\ h_5 &= 15\alpha^2(-3\alpha\beta + 12\beta^2)\varepsilon, \\ h_6 &= (2\alpha^2 - 2\alpha\beta - 4\beta^2) \times [(2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \varepsilon^2(-3\alpha\beta + 12\beta^2))\}], \\ A_3 &= \frac{h_7}{h_8}, \\ h_7 &= 15\alpha^2, \\ h_8 &= 2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2(\omega + \frac{\varepsilon^2(-3\alpha\beta + 12\beta^2)}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma}), \\ A_4 &= (1 + \gamma k_1 + \varepsilon k_2)^2 \eta^i \eta^j + (1 + \gamma k_1 + \varepsilon k_2) \times (\frac{r_7}{r_8} + \frac{r_9}{r_{10}})(\eta^i b^j + \eta^j b^i) + (\frac{r_7}{r_8} + \frac{r_9}{r_{10}})^2 b^i b^j, \\ A_5 &= 1 + (\gamma(1 + \gamma k_1 + \varepsilon k_2) + \varepsilon(\frac{r_7}{r_8} + \frac{r_9}{r_{10}})) \sqrt{\frac{3\alpha - 12\beta}{2\alpha^2 - 2\alpha\beta - 4\beta^2}}, \\ k_1 &= \frac{r_1}{r_2} + \frac{r_3}{r_4}, \\ r_1 &= -3\alpha\beta + 12\beta^2, \quad r_2 = 2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma, \\ r_3 &= 15\alpha^2\varepsilon^2(-3\alpha\beta + 12\beta^2)^2, \\ r_4 &= (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) \times [(2\alpha^2 - 2\alpha\beta - 4\beta^2) \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \varepsilon^2(-3\alpha\beta + 12\beta^2))\}], \end{aligned}$$

$$\begin{aligned}
& 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \\
& \varepsilon^2(-3\alpha\beta + 12\beta^2))\}, \\
& k_2 = \frac{r_5}{r_6}, \\
& r_5 = (-45\alpha\beta + 180\beta^2) \times \alpha^6 \times (2\alpha^2 - 2\alpha\beta - 4\beta^2)^2, \\
& r_6 = [(2\alpha^2 - 2\alpha\beta - 4\beta^2) \times (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + 15\alpha^2\{(\omega \times \\
& (2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma) + \varepsilon^2(-3\alpha\beta + 12\beta^2))\} \times (2\alpha^4 - 2\alpha^3\beta - \\
& 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma)^2], \\
& r_7 = 15\varepsilon\alpha^2, \\
& r_8 = 2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2(\omega + \frac{\varepsilon^2(-3\alpha\beta + 12\beta^2)}{(2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma)}), \\
& r_9 = \varepsilon\gamma(-45\alpha\beta + 180\beta^2)\alpha^2, \\
& r_{10} = \{2\alpha^2 - 2\alpha\beta - 4\beta^2 + 15\alpha^2(\omega + \frac{\varepsilon^2(-3\alpha\beta + 12\beta^2)}{(2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma)})\} \times (2\alpha^4 - \\
& 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma).
\end{aligned}$$

$$\det(a_{ij} + \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)}l_{ilj} + \frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2}b_ib_j + \frac{3\alpha - 12\beta}{2\alpha^2 - 2\alpha\beta - 4\beta^2} \quad (8.4.20)$$

$$\begin{aligned}
\eta_i\eta_j) &= [1 + (\gamma(1 + \gamma k_1 + \varepsilon k_2) + \varepsilon(\frac{r_7}{r_8} + \frac{r_9}{r_{10}}))\sqrt{\frac{3\alpha - 12\beta}{2\alpha^2 - 2\alpha\beta - 4\beta^2}}]r \\
&\times [1 + \omega + \frac{(-3\alpha\beta + 12\beta^2)\varepsilon^2}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma}] \times (1 + 2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + \\
&(-3\alpha\beta + 12\beta^2)\gamma)\det(a_{ij}).
\end{aligned}$$

But $\tilde{g}_{ij} = \frac{1}{2\alpha^6} (6\alpha(\alpha + \beta)^5 - 4(\alpha + \beta)^6) \tilde{H}_{ij}$, with $\tilde{H}_{ij}$ from last step. Thus

$$\tilde{g}^{ji} = \frac{2\alpha^6 \tilde{H}^{ji}}{6\alpha(\alpha + \beta)^5 - 4(\alpha + \beta)^6} \quad (8.4.21)$$

Therefore, from equations (8.4.19) and (8.4.20), we obtain claims (i) and (ii), where $\widetilde{g}_{ij}$ and $\widetilde{g}_{i\bar{j}}$ represent the metric tensors of an $\mathbb{R}$ -Complex Finsler space with cubic (α, β) -metric. $\square$

Observing the terms γ , ε , and δ from the above theorem, we immediately state:

Proposition 8.4.3. *In a non-Hermitian $\mathbb{R}$ -complex Finsler space with cubic (α, β) -metric, the following properties hold:*

$$\gamma + \bar{\gamma} = l_i \eta^i + l_{\bar{j}} \eta^{\bar{j}} = a_{ij} \eta^j \eta^i + a_{\bar{j}\bar{k}} \eta^{\bar{k}} \eta^{\bar{j}} = 2\alpha^2, \quad (8.4.22)$$

$$\varepsilon + \bar{\varepsilon} = b_j \eta^j + b_{\bar{j}} \eta^{\bar{j}} = 2\beta, \quad \delta = \varepsilon, \quad (8.4.23)$$

where,

$$l_i = a_{ij} \eta^j, \quad \eta_i = \frac{1}{2\alpha^6} (6\alpha(\alpha + \beta)^5 - 4(\alpha + \beta)^6) a_{ij} \eta^i + \frac{3(\alpha + \beta)^5}{\alpha^4} b_i, \\ \gamma = a_{jk} \eta^j \eta^k = l_k \eta^k, \quad \varepsilon = b_j \eta^j, \quad b^k = a^{jk} b_j, \quad b_l = b^k a_{kl}, \quad \delta = a_{jk} \eta^j b^k = l_k b^k, \quad l_j = a^{jl} l_i = \eta^j.$$

Furthermore, we present two examples of $\mathbb{R}$ -complex Finsler spaces with cubic (α, β) -metric as follows:

Example 1. We set α as

$$\alpha^2(z, \eta) := \frac{(1 + \varepsilon|z|^2) \sum_{k=1}^{\eta} \operatorname{Re}(\eta^k)^2 - \varepsilon \operatorname{Re} \langle z, \eta \rangle^2}{(1 + \varepsilon|z|^2)^2},$$

where $|z|^2 := \sum_{k=1}^{\eta} z^k \bar{z}^k$, $\langle z, \eta \rangle := \sum_{k=1}^{\eta} z^k \bar{\eta}^k$, defined over the disk

$$\Delta_r^{\eta} = \left\{ z \in C^{\eta}, |z| < r, r := \sqrt{\frac{1}{|\varepsilon|}} \right\}$$

if $\varepsilon < 0$, on C^{η} if $\varepsilon = 0$ and on the complex projective space $P^{\eta}(C)$ if $\varepsilon > 0$.

Through computing, we get $a_{ij} = \frac{1}{1 + \varepsilon|z|^2} \left(\delta_{ij} - \varepsilon \frac{\bar{z}^i \bar{z}^j}{1 + \varepsilon|z|^2} \right)$ and $a_{i\bar{j}} = 0$ and so, $\alpha^2(z, \eta) = \frac{1}{2} (a_{ij} \eta^i \eta^j + a_{\bar{i}\bar{j}} \bar{\eta}^i \bar{\eta}^j)$. Now, taking $\beta(z, \eta) := \operatorname{Re} \frac{\langle z, \eta \rangle}{1 + \varepsilon|z|^2}$, where $b_i := \frac{\bar{z}^i}{1 + \varepsilon|z|^2}$. Here

are a few examples of $\mathbb{R}$ -complex non-Hermitian cubic metrics:

$$F_\varepsilon := \frac{(1 + \varepsilon|z|^2)^2 \left(\frac{\sqrt{(1 + \varepsilon|z|^2) \sum_{k=1}^n \operatorname{Re}(\eta^k)^2 - \varepsilon \operatorname{Re} \langle z, \eta \rangle^2}}{1 + \varepsilon|z|^2} + \operatorname{Re} \frac{\langle z, \eta \rangle}{1 + \varepsilon|z|^2} \right)^3}{(1 + \varepsilon|z|^2) \sum_{k=1}^n \operatorname{Re}(\eta^k)^2 - \varepsilon \operatorname{Re} \langle z, \eta \rangle^2}.$$

Example 2. We set α as

$$\alpha^2(z, \eta) := \frac{|\eta|^2 + \varepsilon(|z|^2|\eta|^2 - |\langle z, \eta \rangle|^2)}{(1 + \varepsilon|z|^2)^2},$$

defined over the disk

$$\Delta_r^\eta = \left\{ z \in C^n, |z| < r, r := \sqrt{\frac{1}{|\varepsilon|}} \right\}$$

if $\varepsilon < 0$, on C^η if $\varepsilon = 0$ and on the complex projective space $P^\eta(C)$ if $\varepsilon > 0$,

where $|\langle z, \eta \rangle|^2 := \langle z, \eta \rangle \overline{\langle z, \eta \rangle}$. Through computing, we get $a_{ij} = 0$ and $a_{i\bar{j}} =$

$\frac{1}{1 + \varepsilon|z|^2} \left(\delta_{i\bar{j}} - \varepsilon \frac{\bar{z}^i z^j}{1 + \varepsilon|z|^2} \right)$ and so, $\alpha^2(z, \eta) = a_{i\bar{j}}(z) \eta^i \bar{\eta}^j$. Setting $\beta(z, \eta)$ as in Example

1., here are a few examples of $\mathbb{R}$ -complex Hermitian cubic metrics:

$$F_\varepsilon := \frac{(1 + \varepsilon|z|^2)^2 \left(\frac{\sqrt{|\eta|^2 + \varepsilon(|z|^2|\eta|^2 - |\langle z, \eta \rangle|^2)}}{1 + \varepsilon|z|^2} + \operatorname{Re} \frac{\langle z, \eta \rangle}{1 + \varepsilon|z|^2} \right)^3}{|\eta|^2 + \varepsilon(|z|^2|\eta|^2 - |\langle z, \eta \rangle|^2)}.$$

CHAPTER 9

On (ρ, β, η) -Metrics with Kropina - Rander Changes

This chapter investigates a novel extension of (ρ, β) -metrics to (ρ, β, η) -metrics by incorporating two 1-forms on a Riemannian manifold. We study the Kropina–Randers metric $F = \frac{\rho^2}{\beta} + \eta$ and establish key results on its metric structure, spray coefficients, projective flatness conditions, and Douglas space characterization, providing deeper insights into the geometric structures within Finsler spaces.

9.1 Introduction

The (ρ, β) -metrics [79] constitute an important and significantly studied class of Finsler metrics, where $\rho = \sqrt{a_{ij}(x)y^i y^j}$ denotes a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form metric. In this context, $a_{ij}(x)$ and $b_i(x)$ depend on the positional coordinates x , and y^i are the components of the tangent vector, representing directional coordinates. These metrics are relatively explicit and computable, and the geometric patterns they exhibit provide a useful foundation for further investigations in Finsler geometry. However, not all Finsler metrics can be expressed in the (ρ, β) -form, which has encouraged researchers to explore more general metric structures to better understand the diversity of geometric phenomena in Finsler spaces. Motivated by this perspective, Yu and Zhu introduced in 2011 the notion of general (ρ, β) -metrics, which are defined by $F = \rho \phi(b^2, u)$, where $\phi = \phi(b^2, u)$ with $u = \frac{\beta}{\rho}$ is a positive smooth function on the interval $(-c_1, c_1)$ and $b = \|\beta\|_\rho$. They investigated fundamental geometric properties of this class, particularly the spray coefficients and the conditions for projective flatness. In 2012, Zhou introduced a general form of spherically symmetric Finsler metrics and characterized the necessary and sufficient conditions for their projective properties and constant flag curvature. This work contributed to a broader understanding of curvature phenomena in symmetric Finsler spaces. Furthermore, Sadeghzadeh and Rajabi proposed a wider class of generalized (ρ, β) -metrics by

introducing an additional 1-form

$$\eta = \eta_i(x)y^i,$$

which is independent of β , on an n -dimensional smooth manifold M . With this extra structure, the metric can be written as

$$F = \rho \Theta(u, \bar{u}), \tag{9.1.1}$$

where ρ is a Riemannian metric and

$$u = \frac{\beta}{\rho}, \quad \bar{u} = \frac{\eta}{\rho}.$$

The function $\Theta(u, \bar{u})$ is assumed to be of the form

$$\Theta(u, \bar{u}) = \bar{u} \phi(u) \Xi\left(\frac{\bar{u}}{\bar{u} \phi(u)}\right),$$

subject to the conditions $\|\beta\|_\rho < l_1$ and $\|\eta\|_\rho < l_1$. Here $\bar{u} \phi$ and Θ are positive C^∞ functions defined on $(-c_1, c_1) \times (-l_1, l_1)$.

This construction is called a (ρ, β, η) -metric. In general, it does not reduce to a classical (ρ, β) -metric, so it strictly enlarges that category. It provides a unified framework that includes both generalized (ρ, β) -metrics and spherically symmetric Finsler metrics. Several geometric aspects of this family, such as projective flatness and the characterization of Douglas-type spaces, have been investigated. Moreover, this metric can be interpreted as arising from a β -deformation of a Finsler structure.

9.2 Preliminaries

Let M be a smooth manifold and denote its tangent bundle by

$$TM = \bigcup_{x \in M} T_x M,$$

where $T_x M$ is the tangent space at $x \in M$. A Finsler structure on M is a function

$$F : TM \longrightarrow [0, \infty)$$

satisfying the following conditions:

1. F is C^∞ on $TM \setminus \{0\}$;
2. $F(x, \lambda y) = \lambda F(x, y)$ for every $\lambda > 0$;
3. For each $x \in M$, the symmetric bilinear form

$$g_y(a, b) = \frac{1}{2} \frac{\partial^2}{\partial r \partial s} \left[F^2(y + ra + sb) \right] \Big|_{r=s=0}, \quad a, b \in T_x M,$$

is positive definite.

To measure the deviation of $F_x := F|_{T_x M}$ from a Euclidean norm, define the Cartan tensor by

$$\mathcal{C}_y(a, b, c) = \frac{1}{2} \frac{d}{dt} [g_{y+tc}(a, b)] \Big|_{t=0}, \quad a, b, c \in T_x M.$$

The collection $\{\mathcal{C}_y\}_{y \in TM \setminus \{0\}}$ is called the Cartan torsion. It vanishes entirely if and only if F is Riemannian.

In a Finsler manifold (M, F) , the metric induces a canonical spray vector field on $TM \setminus \{0\}$.

The local coordinates (x^i, y^i) are given by [143]

$$G = y^j \frac{\partial}{\partial x^j} - 2G^i(x, y) \frac{\partial}{\partial y^i},$$

where

$$G^i = \frac{1}{4} g^{im} \left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m} \right) y^j y^k.$$

This vector field is called the spray associated with F , and the geodesics on M are obtained as the projections of its integral curves.

Finally, let $F(x, y)$ be a Finsler metric defined on an open subset $U \subset \mathbb{R}^n$. The metric is said to be projectively flat if all of its geodesics are straight lines in U . This property holds if and only if

$$F_{x^k y^j} y^k - F_{x^j} = 0.$$

In this case, the spray coefficients satisfy

$$G^i = Q(x, y) y^i, \quad Q = \frac{F_{x^k y^k}}{2F}.$$

The projective and locally projectively flat properties of various Finsler metrics have been extensively studied by several authors [19].

In this research work, we further investigate (ρ, β, η) -metrics, which are positively homogeneous functions of degree one in ρ , β and η , defined on an n -dimensional manifold M in the form given by equation (9.1.1).

If $\phi(u)$ is taken to be a homogeneous function of degree zero in ρ and β , then the metric in (9.1.1) reduces to

$$F = \frac{\rho^2}{\beta} + \eta,$$

which is of Kropina–Randers type.

9.3 (ρ, β, η) - Metrics for a Kropina-Randers Change

Definition 9.3.1. [143] Let ρ be a Riemannian metric and β, η be two 1-forms on an n -dimensional manifold M , then an (ρ, β, η) -metric F is defined by

$$F = \rho \Theta(u, \bar{u}), \quad u := \frac{\beta}{\rho}, \quad \bar{u} = \frac{\eta}{\rho}$$

where $\|\beta\|_\rho < c_1$, $\|\eta\|_\rho < c_2$, and $\Theta(u, \bar{u})$ is a positive C^∞ function on the interval $(-c_1, c_1) \times (-l_1, l_1)$.

if describing a property For (ρ, β, η) -metrics, the Kropina–Randers change is defined as

$$\bar{F} = F + \eta,$$

where,

$$F = \frac{\rho^2}{\beta} + \eta.$$

The fundamental tensor of a (ρ, β, η) -metric under a Kropina–Randers change can be expressed as follows:

$$\begin{cases} g_{ij} = \sigma a_{ij} + \sigma_0 b_i b_j + \bar{\sigma}_0 \eta_i \eta_j + \sigma_1 (b_i \rho_j + b_j \rho_i) + \bar{\sigma}_1 (\eta_i \rho_j + \eta_j \rho_i) + \sigma_2 \rho_i \rho_j + \\ \sigma_3 (b_i \eta_j + b_j \eta_i), \end{cases} \quad (9.3.1)$$

where,

$$\begin{cases} \sigma = (\frac{1}{u} + \bar{u})(\frac{2}{u}), \quad \sigma_0 = \frac{3}{u^4} + \frac{2\bar{u}}{u^3}, \quad \sigma_1 = \frac{-4}{u^3} - \frac{2\bar{u}}{u^2}, \quad \sigma_2 = \frac{4}{u^2}, \quad \bar{\sigma}_0 = 1, \quad \bar{\sigma}_1 = \frac{2}{u}, \\ \sigma_3 = \frac{-1}{u^2}. \end{cases} \quad (9.3.2)$$

Moreover,

$$\det(g_{ij}) = \left(\frac{1}{u} + \bar{u}\right)^{n+1} \left(\frac{2}{u}\right)^{n-2} \Gamma \det(a_{ij}). \quad (9.3.3)$$

Where,

$$\Gamma = \frac{2}{u} + (b^2 - u^2) \frac{2}{u^3}, \quad (9.3.4)$$

and

$$b^2 = a^{ij} b_i b_j, \quad g^2 := a^{ij} \eta_i \eta_j, \quad \theta := a^{ij} b_i \eta_j, \quad J := 0.$$

$$g^{ij} = \frac{1}{\sigma} \left\{ a^{ij} - \frac{1}{\Gamma} [\Theta_{uu} + (g^2 - \bar{u}^2)J] b^i b^j - \frac{1}{\Gamma} [\Theta_{\bar{u}\bar{u}} + (b^2 - u^2)J] \eta^i \eta^j \right. \\ \left. - \frac{1}{\Theta\Gamma} [\sigma_1 + \Theta_2(\theta - u\bar{u}) - \Theta_1(g^2 - \bar{u}^2)] (b^i \rho^j + b^j \rho^i) - \frac{1}{\Theta\Gamma} [\bar{\sigma}_1 - \Theta_2(b^2 - u^2) + \right. \\ \left. \Theta_1(\theta - u\bar{u})] (\eta^i \rho^j + \eta^j \rho^i) + \frac{1}{\Theta^2\Gamma} (u\Theta + (b^2 - u^2)\Theta_u + (\theta - u\bar{u})\Theta_{\bar{u}} \right. \\ \left. + [\sigma_1 + \Theta_2(\theta - u\bar{u}) - \Theta_1(g^2 - \bar{u}^2)]) + [\bar{u}\Theta + (g^2 - \bar{u})\Theta_{\bar{u}} + (\theta - u\bar{u})\Theta_u \right. \\ \left. + [\bar{\sigma}_1 - \Theta_2(b^2 - u^2) + \Theta_1(\theta - u\bar{u})] \rho^i \rho^j \right\}, \quad (9.3.5)$$

where,

$$\Theta_1 = 0, \quad \Theta_2 = \frac{-2}{u^3}, \quad \Theta = \frac{1}{u} + \bar{u}, \quad \Theta_u = \frac{-1}{u^2}, \quad \Theta_{uu} = \frac{2}{u^3}, \quad \Theta_{\bar{u}} = 1, \quad \Theta_{\bar{u}\bar{u}} = 0, \\ \Theta_{u\bar{u}} = 0.$$

Additionally, the Cartan tensor of F is given by

$$C_{ijk} = \left(\frac{-2}{u^3} - \frac{\bar{u}}{u^2} \right) [(b_k - u\rho_k)\rho_{ij} + (b_i - u\rho_i)\rho_{jk} + (b_j - u\rho_j)\rho_{ik}] + \frac{1}{u} \\ \left[(\eta_k - \bar{u}\rho_k)\rho_{ij} + (\eta_i - \bar{u}\rho_i)\rho_{jk} + (\eta_j - \bar{u}\rho_j)\rho_{ik} \right] + \frac{1}{2\rho} \left(\frac{3}{u^4} + \frac{2\bar{u}}{u^3} \right) \bar{u} \\ \left[(b_i - u\rho_i)(b_j - u\rho_j)(\eta_k - \bar{u}\rho_k) + (b_j - u\rho_j)(b_k - u\rho_k)(\eta_i - \bar{u}\rho_i) \right. \\ \left. + (b_i - u\rho_i)(b_k - u\rho_k)(\eta_j - \bar{u}\rho_j) \right] + \frac{\bar{u}}{2\rho} \left[(\eta_i - \bar{u}\rho_i)(\eta_j - \bar{u}\rho_j)(b_k - u\rho_k) \right. \\ \left. + (\eta_j - \bar{u}\rho_j)(\eta_k - \bar{u}\rho_k)(b_i - u\rho_i) + (\eta_i - \bar{u}\rho_i)(\eta_k - \bar{u}\rho_k)(b_j - u\rho_j) \right] \\ + \frac{u}{2\rho} \left(\frac{3}{u^4} + \frac{2\bar{u}}{u^3} \right) (b_i - u\rho_i)(b_j - u\rho_j)(b_k - u\rho_k) + \frac{\bar{u}}{2\rho} [(\eta_i - \bar{u}\rho_i)(\eta_j - \bar{u}\rho_j) \\ (\eta_k - \bar{u}\rho_k)].$$

Proposition 9.3.2. *The fundamental tensor for an (ρ, β, η) -metric for a Kropina–Randers Change can be also represented as follows:*

$$\left\{ \begin{aligned} g_{ij} &= \sigma \left\{ a_{ij} + \delta_1 b_i b_j + \delta_2 \eta_i \eta_j + \delta_0 (b_i + \eta_i)(b_j + \eta_j) + \frac{\sigma_2}{\sigma} (\rho_i + \frac{\sigma_1}{\sigma_2} b_i + \frac{\bar{\sigma}_1}{\sigma_2} \eta_i) \right. \\ &\quad \left. (\rho_j + \frac{\sigma_1}{\sigma_2} b_j + \frac{\bar{\sigma}_1}{\sigma_2} \eta_j) \right\}. \end{aligned} \right.$$

Where,

$$\sigma = \left(\frac{1}{u} + \bar{u} \right) \left(\frac{2}{u} \right), \quad \sigma_1 = \frac{-4}{u^3} - \frac{2\bar{u}}{u^2}, \quad \sigma_2 = \frac{4}{u^2}, \quad \bar{\sigma}_1 = \frac{2}{u}, \quad \delta_0 = \frac{u^2}{2(1+u\bar{u})} \left(\frac{-1}{u^2} - \frac{-2(-4-2u\bar{u})}{4u^2} \right),$$

$$\delta_1 = \frac{u^2}{2(1+u\bar{u})} \left(\frac{3u+2\bar{u}}{u^4} - \frac{(-4-2u\bar{u})^2}{4u^4} \right) - \frac{u^2}{2(1+u\bar{u})} \left(\frac{-1}{u^2} - \frac{2(-4-2u\bar{u})}{4u^2} \right), \quad \delta_2 = \frac{-u^2}{2(1+u\bar{u})} \left(\frac{-1}{u^2} - \frac{2(-4-2u\bar{u})}{4u^2} \right).$$

Proof. The fundamental tensor and Cartan tensor of a Finsler metric F are defined as

$$g_{ij} = \frac{1}{2} [F^2]_{y^i y^j} = F F_{y^i y^j} + F_{y^i} F_{y^j}$$

and

$$C_{ijk} = \frac{1}{2} (g_{ij})_{y^k},$$

respectively. A direct calculation yields

$$u_{y^i} = \frac{b_i - u\rho_i}{\rho}, \quad \text{where } \rho_i = \rho_{y^i},$$

$$\bar{u}_{y^i} = \frac{(\eta_i - \bar{u}\rho_i)}{\rho},$$

$$(\Theta)_{y^i} = \frac{1}{\rho} \left[\frac{-h_i}{u^2} - \bar{h}_i \right],$$

$$(\Theta_u)_{y^i} = \frac{1}{\rho} \left[\frac{2h_i}{u^3} \right],$$

$$(\Theta_{\bar{u}})_{y^i} = 0,$$

$$(h_i)_{y^j} = \frac{-1}{\rho} (b_j - u\rho_j)\rho_i - u\rho_{ij}, \quad \text{where } \rho_{ij} = \rho_{y^i y^j} = \frac{1}{\rho} (a_{ij} - \rho_i \rho_j),$$

$$(\bar{h}_i)_{y^j} = \frac{-1}{\rho} (\eta_j - \bar{u}\rho_j)\rho_i - \bar{u}\rho_{ij}.$$

Let l_i be defined as F_y^i and l_{ij} as $F_{y^i y^j}$. From the previous equations, we obtain

$$l_i = \left(\frac{1}{u} + \bar{u} \right) \rho_i + \left(\frac{-1}{u^2} \right) h_i + \bar{h}_i, \quad (9.3.6)$$

$$l_{ij} = \left(\frac{2}{u} \right) \rho_{ij} + \frac{2h_i h_j}{\rho u^3}. \quad (9.3.7)$$

This leads us to (9.3.1). We can express (9.3.1) in a different way as follows:

$$\begin{cases} g_{ij} = \sigma \left\{ a_{ij} + \delta_1 b_i b_j + \delta_2 \eta_i \eta_j + \delta_0 (b_i + \eta_i)(b_j + \eta_j) + \frac{\sigma_2}{\sigma} \left(\rho_i + \frac{\sigma_1}{\sigma_2} b_i + \frac{\bar{\sigma}_1}{\sigma_2} \eta_i \right) \right. \\ \left. \left(\rho_j + \frac{\sigma_1}{\sigma_2} b_j + \frac{\bar{\sigma}_1}{\sigma_2} \eta_j \right) \right\}. \end{cases}$$

Where,

$$\sigma = (\frac{1}{u} + \bar{u})(\frac{2}{u}), \quad \sigma_1 = \frac{-4}{u^3} - \frac{2\bar{u}}{u^2}, \quad \sigma_2 = \frac{4}{u^2}, \quad \bar{\sigma}_1 = \frac{2}{u}, \quad \delta_0 = \frac{u^2}{2(1+u\bar{u})}(\frac{-1}{u^2} - \frac{-2(-4-2u\bar{u})}{4u^2}),$$

$$\delta_1 = \frac{u^2}{2(1+u\bar{u})}(\frac{3u+2\bar{u}}{u^4} - \frac{(-4-2u\bar{u})^2}{4u^4}) - \frac{u^2}{2(1+u\bar{u})}(\frac{-1}{u^2} - \frac{2(-4-2u\bar{u})}{4u^2}), \quad \delta_2 = \frac{-u^2}{2(1+u\bar{u})}(\frac{-1}{u^2} - \frac{2(-4-2u\bar{u})}{4u^2}).$$

By applying [2, Lemma 1.1.1] four times, we derive (9.3.3) and (9.3.5). $\square$

Note: It is clear that the previously mentioned proposition is valid for any (ρ, β) -metric by choosing $\bar{u} = 0$, and it is also valid for any (ρ, η) -metric by setting $u = 0$. The Cauchy-Schwarz inequality states that $\theta^2 = (a^{ij}b_i\eta_j)^2 \leq (a^{ij}b_ib_j)(a^{ij}\eta_i\eta_j) = b^2g^2$. We must prove the validity of the following statement.

Proposition 9.3.3. *Let M symbolize an n -dimensional manifold. A Finsler metric F is defined as $F = \rho\Theta(s, \bar{u})$, where $\Theta(u, \bar{u}) = \frac{1}{u} + \bar{u}$, with $u = \frac{\beta}{\rho}$ and $\bar{u} = \frac{\eta}{\rho}$. This metric applies to Riemannian ρ and 1-forms $\beta = b_i y^i$ and $\eta = \eta_i y^i$ under the conditions that $\|\beta\|_\rho < c_1$ and $\|\eta\|_\rho < c_2$. The condition $\theta - u\bar{u} \geq 0$ is only valid if the corresponding positive C^∞ function $\Theta = \Theta(u, \bar{u})$ is fulfilled.*

$$\Theta := \frac{2}{u} > 0, \quad \eta > 0. \quad (9.3.8)$$

When $n \geq 3$ or $\eta > 0$, for $n = 2$, with η defined by (9.3.4) and $u, \bar{u}, b, g$ being arbitrary values satisfying $|u| \leq b < c_1$ and $|\bar{u}| \leq g < c_2$, the result holds.

Proof. The Case when $n = 2$ is similar to the scenarios where $n \geq 3$, so we will prove the proposition only for $n \geq 3$. It is easy to verify that F is a function demonstrating regularity and positive homogeneity. We will now analyze the condition for strong convexity.

Assuming that (9.3.8) holds, we can express $\Theta\eta$ as a quadratic equation in terms of Θ in the following manner.

$$\Theta\eta = \frac{4}{u^2} + (a + \bar{a})\frac{2}{u} + (a\bar{a} - b\bar{b}) > 0, \quad (9.3.9)$$

where

$$a := \frac{2(b^2 - u^2)}{u^3},$$

$$b := 0,$$

$$\bar{a} := 0,$$

$$\bar{b} := \frac{2(\theta - u\bar{u})}{u^3}.$$

□

The inequality stated above is true if and only if one of the following conditions is satisfied.

1.

$$\Delta < 0 \text{ where } \Delta = \left(\frac{2(b^2 - u^2)}{u^3}\right)^2;$$

2.

$$\Delta = 0, \text{ then } \Theta \neq w \text{ and } \Theta\Gamma = (\Theta - w)^2 \text{ where } w = \frac{-1}{2}\left(\frac{2(b^2 - u^2)}{u^3}\right);$$

3.

$$\Delta > 0, \text{ then } 0 < \Theta < w_1 \text{ or } \Theta > w_2 \text{ where } w_1 := \frac{-1}{2} \left[\left(\frac{2(b^2 - u^2)}{u^3}\right) + \sqrt{\Delta} \right]$$

$$\text{and } w_2 := \frac{-1}{2} \left[\left(\frac{2(b^2 - u^2)}{u^3}\right) - \sqrt{\Delta} \right].$$

Take note that w_1 is less than w_2 .

Let us take an assortment of functions $\Theta_f(u, \bar{u}) = 1 - f + f\left(\frac{1}{u^2\bar{u}}\right)$, $0 \leq f \leq 1$. Put $F_f = \rho\Theta_f(u, \bar{u})$ and $g_{ij}^f = \frac{1}{2}[F_f^2]_{y^i y^j}$, then $F_0 = \rho$ and $F_1 = F$. For every $0 \leq f \leq 1$, $|u| \leq b < c_1$ and $|\bar{u}| \leq g < l_1$, we will demonstrate $\Theta_f > 0$ and $\Gamma_f > 0$. It is evident that $\Theta_f = 1 - f + f\Theta > 0$. Moreover, $\Theta_f\Gamma_f = \Theta_f^2 + f(a + \bar{a})\Theta_f + f^2(a\bar{a} - b\bar{b})$. Then we have $\Delta_f = f^2\Delta$ where

$$\Delta = \left(\frac{2(b^2 - u^2)}{u^3}\right)^2. \quad (9.3.10)$$

It is clear that when $\Delta_f(u, \bar{u}) < 0$, the expression $\Theta_f \Gamma_f$ is strictly positive, meaning that Γ_f is greater than zero.

Let us consider the scenario where there exist values f_0 and $\Delta_{f_0}(u_0, \bar{u}_0) > 0$. Given that $\Delta_f(u, \bar{u})$ is continuous with respect to both f and $(u, \bar{u})$, it follows that there exists a set $D \subset (-c_1, c_1) \times (-c_2, c_2)$ such that for every $(u, \bar{u}) \in D$, $\Delta_f(u, \bar{u}) > 0$, and for all $(u, \bar{u}) \in \partial D$, $\Delta_f(u, \bar{u}) = 0$, where ∂D denotes the boundary of D . Consequently, within D , we find that

$$\Theta_f \Gamma_f = (\Theta_f - f w_1)(\Theta_f - f w_2). \quad (9.3.11)$$

If we have $\Gamma_f(u, \bar{u}) > 0$ on domain D , then there is nothing more to demonstrate. Assume there is a subset $v \subset D$ such that for $(u, \bar{u}) \in v = u \cup \partial u$, $\Gamma_f(u, \bar{u}) \leq 0$. Given that both Γ_0 and η_1 are positive, we can invoke continuity for Γ_f , which yields $\exists f_1, f_2 \in (0, 1)$ such that $\Gamma_{f_1}(u, \bar{u}) = \Gamma_{f_2}(u, \bar{u}) = 0 \forall (u, \bar{u}) \in v$. According to (9.3.11), we have

$$(\Theta_{f_1} - f_1 w_1)(\Theta_{f_1} - f_1 w_2) = 0, \text{ and } (\Theta_{f_2} - f_2 w_1)(\Theta_{f_2} - f_2 w_2) = 0. \quad (9.3.12)$$

For $f_1 \leq f \leq f_2$, it follows that for all $(u, \bar{u}) \in v$, $\Gamma_f(u, \bar{u}) \leq 0$, while for every $(u, \bar{u}) \in D - v$, $\Gamma_f(u, \bar{u}) > 0$. Due to the continuity of Γ_f , we find that $\Gamma_f(u, \bar{u}) = 0$ for $f_1 \leq f \leq f_2$, whenever $(u, \bar{u}) \in \partial v$. The equation (9.3.11) gives us $\Theta_f = f w_1$ or $\Theta_f = f w_2$. Consequently, from (9.3.12), we arrive at $f_1 = f_2$, which leads to a contradiction. Thus, it must be the case that $\Gamma_f(u, \bar{u}) > 0$ on D .

Now let there is $D_1 \subset (-c_1, c_1) \times (-c_2, c_2)$ such that $\Delta(u, \bar{u}) = 0$ for every $(u, \bar{u}) \in D_1$. Then we see that for every $0 \leq f \leq 1$ and $(u, \bar{u}) \in D_1$ we have $\Delta_f(u, \bar{u}) = 0$. One could easily get $\Theta_f \Gamma_f - f^2 \Theta \Gamma = (1 - f)(1 - f + 2f(\Theta + \frac{m+\bar{m}}{2}))$. If for some $0 < f < 1$ we have $(1 - f + 2f(\Theta + \frac{m+\bar{m}}{2})) \geq 0$ then $\Theta_f \Gamma_f \geq f^2 \Theta \Gamma > 0$ and therefore $\Gamma_f > 0$. Now we assume that there are $0 < f < 1$ such that

$$1 - f + 2f(\Theta + \frac{a+\bar{a}}{2}) < 0. \quad (9.3.13)$$

Which one could easily get $1 - f + f(\Theta - \frac{m+\bar{m}}{2})) < \frac{1}{2}(1 - f) \neq 0$. Thus

$$\Theta_f \Gamma_f = (\Theta_f - w_f)^2 = (1 - f + f(\Theta - w))^2 = (1 - f + f(\Theta - \frac{a+\bar{a}}{2}))^2 > 0. \quad (9.3.14)$$

Then for this $0 < f < 1$ we get $\Gamma_f > 0$, too.

All the arguments presented indicate that $\Gamma_f > 0$ for every $0 \leq f \leq 1$. Consequently, we have $\det(g_{ij}^f) > 0$ for all values of $0 \leq f \leq 1$. Given that (g_{ij}^0) is positive definite, we can deduce that (g_{ij}^f) remains positive definite for all $f \in [0, 1]$. As a result, F_f qualifies as a Finsler metric for any $f \in [0, 1]$.

Assuming that $F = \rho\Theta(u, \bar{u})$ and that $\det(g_{ij})$ is positive, it follows from Proposition (9.3.2) that the condition $\det(g_{ij}) > 0$ is equivalent to $\Theta^{n-2}\Gamma > 0$, which indicates that $\Theta \neq 0$ when $n \geq 3$. Using $\Theta(0,0) > 0$, we may construct the inequality $\Theta > 0$. Furthermore, $\det(g_{ij}) > 0$ implies that $\Gamma > 0$.

For 1-form $\beta = b_i(x)y^i$ and $\eta = \eta_i(x)y^i$, we have [143]

$$\beta_{rij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad \beta_{u_{ij}} := \frac{1}{2}(b_{i|j} - b_{j|i}). \quad (9.3.15)$$

$$\eta_{rij} := \frac{1}{2}(\eta_{i|j} + \eta_{j|i}), \quad \eta_{u_{ij}} := \frac{1}{2}(\eta_{i|j} - \eta_{j|i}). \quad (9.3.16)$$

9.4 Spray Coefficients of F

In this part, we employ a method introduced by Matsumoto to calculate G^i . For $F = \rho\Theta(u, \bar{u})$, we obtain

$$\beta_{x^j} = b_{0|j} + b_r G_j^r, \quad \eta_{x^j} = \eta_{0|j} + \eta_r G_j^r,$$

$$u_{x^j} = \frac{1}{\rho}(b_{0|j} + h_r G_j^r), \quad \bar{u}_{x^j} = \frac{1}{\rho}(\eta_{0|j} + \bar{h}_r G_j^r), \quad (9.4.1)$$

where $G_j^i = {}^\rho G_{y^j}^i$. Moreover, by $\rho_{|i} = 0$ and $\rho_{i|j} = 0$ we have

$$\rho_{x^j} = \rho_r G_j^r, \quad (\rho_i)_{x^j} = \rho_{ir} G_j^r + \rho_r G_{ij}^r, \quad (9.4.2)$$

where $G_{ij}^r = {}^\rho G_{y^i y^j}^r$. Then

$$\begin{aligned} (h_i)_{x^j} &= b_{i|j} - \frac{1}{\rho} b_{0|j} \rho_i - \frac{1}{\rho} h_r G_j^r \rho_i + h_r G_{ij}^r - u \rho_{ir} G_j^r, \\ (\bar{h}_i)_{x^j} &= \eta_{i|j} - \frac{1}{\rho} \eta_{0|j} \rho_i - \frac{1}{\rho} \bar{h}_r G_j^r \rho_i + \bar{h}_r G_{ij}^r - \bar{u} \rho_{ir} G_j^r. \end{aligned} \quad (9.4.3)$$

By differentiating (9.3.6) with respect to x^j and employing (9.4.1), (9.4.2), and (9.4.3), we obtain the following results:

$$\left\{ \begin{aligned} \frac{\partial l_i}{\partial x^j} &= \left(\frac{-1}{u^2}\right) b_{i|j} + \eta_{i|j} + \frac{1}{\rho} \left(\frac{2}{u^3} b_{0|j}\right) h_i + \left(\left(\frac{1}{u} + \bar{u}\right) \rho_r - \left(\frac{1}{u^2}\right) h_r + \bar{h}_r\right) G_{ij}^r \\ &\quad + \left(\frac{2}{u}\right) \rho_{ir} G^r + \frac{1}{\rho} \left(\frac{2}{u^3} h_i h_r\right) G_j^r. \end{aligned} \right.$$

Let $\mathcal{D}$ denote the horizontal covariant derivative with respect to the Cartan connection of F .

We now assume $l_{i;j} = 0$, meaning that $\frac{\partial l_i}{\partial x^j} = l_{ir} N_j^r + l_r \eta_{ij}^r$. For our analysis, let us define

$$\mathcal{D}_{jk}^i := \Gamma_{jk}^i - G_{jk}^i, \quad \mathcal{D}_j^i := \mathcal{D}_{jk}^i y^k = N_j^i - G_j^i, \quad \mathcal{D}^i := \mathcal{D}_j^i y^j = 2G^i - 2{}^\rho G^i. \quad (9.4.4)$$

Then $\frac{\partial l_i}{\partial x^j} = l_{ir}(\mathcal{D}_j^r + G_j^r) + l_r(\mathcal{D}_{ij}^r + G_{ij}^r)$. Substituting (9.3.6) and (9.3.7) into the above

equation yields

$$\left\{ \begin{array}{l} \frac{\partial l_i}{\partial x^j} = \left[\left(\frac{2}{u} \right) \rho_{ir} + \frac{2h_i h_r}{\rho u^3} \right] D_j^r + \left(\left(\frac{1}{u} + \bar{u} \right) \rho_r + \left(\frac{-1}{u^2} \right) (h_i) \right) D_{ij}^r + \left(\left(\frac{1}{u} + \bar{u} \right) \rho_r \right. \\ \left. - \left(\frac{1}{u^2} \right) h_r + \bar{h}_r \right) G_{ij}^r + \left(\left(\frac{2}{u} \right) \rho_{ir} + \frac{1}{\rho} \left(\frac{2}{u^3} \right) h_i h_r \right) G_j^r. \end{array} \right. \quad (9.4.5)$$

By examining (9.4.4) and (9.4.5), we can derive the following

$$\left\{ \begin{array}{l} \left(\frac{-1}{u^2} \right) b_{i|j} + \eta_{i|j} = \left(\left(\frac{2}{u} \right) \rho_{ir} + \frac{2h_i h_r}{\rho u^3} \right) D_j^r + \left(\left(\frac{1}{u} + \bar{u} \right) \rho_r + \left(\frac{-1}{u^2} \right) (b_r - u \rho_r) + (\eta_r - \bar{u} \rho_r) \right) D_{ij}^r - \frac{1}{\rho} \left(\frac{2}{u^3} \right) b_{0|j} h_i. \end{array} \right. \quad (9.4.6)$$

Therefore, using (9.3.15) and (9.3.16), we obtain

$$\left\{ \begin{array}{l} \frac{-2}{u^2} \beta_{rij} + 2\eta_{rij} = D_j^r \left(\left(\frac{2}{u} \right) \rho_{ir} + \frac{2h_i h_r}{\rho u^3} \right) + D_i^r \left(\left(\frac{2}{u} \right) \rho_{jr} + \frac{2h_j h_r}{\rho u^3} \right) + 2 \left(\left(\frac{1}{u} + \bar{u} \right) \rho_r \right. \\ \left. + \left(\frac{-1}{u^2} \right) (b_r - u \rho_r) + (\eta_r - \bar{u} \rho_r) \right) D_{ij}^r - \frac{1}{\rho} \left(\frac{2}{u^3} \right) b_{0|j} h_i - \frac{1}{\rho} \left(\frac{2}{u^3} \right) b_{0|i} h_j. \end{array} \right. \quad (9.4.7)$$

$$\left\{ \begin{array}{l} \frac{-2}{u^2} \beta_{sij} + 2\eta_{sij} = D_j^r \left(\left(\frac{2}{u} \right) \rho_{ir} + \frac{2h_i h_r}{\rho u^3} \right) + D_i^r \left(\left(\frac{2}{u} \right) \rho_{jr} + \frac{2h_j h_r}{\rho u^3} \right) - \frac{1}{\rho} \left(\frac{2}{u^3} \right) b_{0|j} h_i + \\ \frac{1}{\rho} \left(\frac{2}{u^3} \right) b_{0|i} h_j. \end{array} \right. \quad (9.4.8)$$

By contracting (9.4.7) and (9.4.8) with y^j , we can conclude that

$$\left\{ \begin{array}{l} \frac{-2}{u^2} \beta_{ri0} + 2\eta_{ri0} = D^r \left(\left(\frac{2}{u} \right) \rho_{ir} + D_i^r \left(\left(\frac{2}{u} \right) \rho_{jr} + \frac{2h_j h_r}{\rho u^3} \right) + 2 \left(\left(\frac{1}{u} + \bar{u} \right) \rho_r \right. \right. \\ \left. \left. + \left(\frac{-1}{u^2} \right) (b_r - u \rho_r) + (\eta_r - \bar{u} \rho_r) \right) D_i^r - \frac{1}{\rho} \left(\frac{2}{u^3} \right) \beta_{r00} h_i. \end{array} \right. \quad (9.4.9)$$

$$\left\{ \begin{array}{l} \frac{-2}{u^2} \beta_{ui0} + 2\eta_{ui0} = D^r \left(\left(\frac{2}{u} \right) \rho_{ir} + \frac{2h_i h_r}{\rho u^3} \right) - \frac{1}{\rho} \left(\frac{2}{u^3} \right) \beta_{r00} h_i. \end{array} \right. \quad (9.4.10)$$

Subtracting (9.4.10) from (9.4.9), we obtain

$$\left\{ \begin{array}{l} \left(\frac{-1}{u^2} \right) (\beta_{ri0} - \beta_{si0}) + (\eta_{ri0} - \eta_{si0}) = \left\{ \left(\frac{1}{u} + \bar{u} \right) \rho_i + \left(\frac{-1}{u^2} \right) (b_r - u \rho_r) + \right. \\ \left. (\eta_r - \bar{u} \rho_r) \right\} D_i^r \end{array} \right. \quad (9.4.11)$$

The contraction of (9.4.11) with y^i leads to

$$\left\{ \frac{-1}{u^2}(\beta_{r00}) + \eta_{r00} = \left(\left(\frac{1}{u} + \bar{u} \right) \rho_i - \left(\frac{1}{u^2} \right) (b_r - u \rho_r) + (\eta_r - \bar{u} \rho_r) \right) D^r. \right. \quad (9.4.12)$$

In order to determine the spray coefficients of F , we first establish the following lemma.

Lemma 9.4.1. *The algebraic equation system (i) $l_{ir}A^r = B_i$ and (ii) $l_rA^r = B$ has a distinct solution for A^r when B and B_i are defined, with the condition that $B_i y^i = 0$. The solution may be stated as*

$$A^i = (\rho_r A^r) \rho^i + \frac{\rho}{\Theta} B^i - \frac{\rho}{\Theta \Gamma} (\mu_1 h^i + \mu_2 \bar{h}^i), \quad (9.4.13)$$

where,

$$B^i = a^{il} B_l, \quad h^i = a^{il} h_l, \quad \bar{h}^i = a^{il} \bar{h}_l \quad \text{and}$$

$$\Theta := \frac{2}{u},$$

$$\mu_1 := \left(\frac{2}{u^3} \right) B_r b^r,$$

$$\mu_2 := 0.$$

Proof. Contracting (9.3.7) with b^i and η^i , we obtain

$$\left\{ l_{ij} b^i = \frac{1}{\rho} \left[\frac{2}{u} + (b^2 - u^2) \frac{2}{u^3} \right] h_j. \right. \quad (9.4.14)$$

$$\left\{ l_{ij} \eta^i = \frac{1}{\rho} \left[(\theta - u\bar{u}) \frac{2}{u^3} \right] h_j + \frac{1}{\rho} \left[\left(\frac{2}{u} \right) \right] \bar{h}_j. \right. \quad (9.4.15)$$

Next, contracting (i) with b^i and η^i and using (9.4.14) and (9.4.15), we obtain

$$\left\{ \begin{array}{l} \left[\frac{2}{u} + (b^2 - u^2) \frac{2}{u^3} \right] h_j A^j = \rho B_j b^j, \\ \left[(\theta - u\bar{u}) \frac{2}{u^3} \right] h_j A^j + \left[\frac{2}{u} \right] \bar{h}_j A^j = \rho B_j \eta^j. \end{array} \right.$$

By solving the above system, we obtain

$$\left\{ h_j A^j = \frac{\rho}{\Theta \eta} \left\{ \frac{2}{u} \right\} B_j b^j, \right. \quad (9.4.16)$$

$$\left\{ \bar{h}_j A^j = \frac{\rho}{\Theta \bar{\eta}} \left\{ \frac{2}{u} + (b^2 - u^2) \frac{2}{u^3} \right\} B_j \eta^j - \left\{ (\theta - u\bar{u}) \frac{2}{u^3} \right\} B_j b^j. \right. \quad (9.4.17)$$

Substituting (9.3.6) in equation (ii) yields $\Theta \rho_j A^j + \Theta_u h_j A^j + \Theta_{\bar{u}} \bar{h}_j A^j = B$. By (9.4.16) and (9.4.17), we get

$$\left\{ \rho_j A^j = \frac{1}{(\frac{1}{u} + \bar{u})} \left\{ B - \frac{\rho}{\Theta \bar{\eta}} \left(\frac{-1}{u^2} \left\{ \frac{2}{u} - (\theta - u\bar{u}) \frac{2}{u^3} \right\} \right) B_j b^j - \frac{\rho}{\Theta \bar{\eta}} \left(\frac{2}{u} + (b^2 - u^2) \frac{2}{u^3} \right) B_j \eta^j. \right. \right.$$

Substituting (9.3.7) into equation (i) yields

$$\left\{ \frac{2}{u\rho} [a_{ij} A^j - (\rho_j A^j) \rho_i] + \frac{1}{\rho} \left\{ \left[\left(\frac{2}{u^3} \right) h_i \bar{h}_j A^j \right] \right\} = B_i. \right.$$

By contracting this equation with a^{ij} and using (9.4.16) and (9.4.17), one can derive (9.4.13).

At this point, we can determine the spray coefficients of F .

The system of algebraic equations represented by (9.4.10) and (9.4.12) has a solution, given by Lemma (9.4.1), expressed as $D^i = (\rho_r D^r) \rho^i + \frac{\rho}{\Theta} B^i - \frac{\rho}{\Theta \bar{\eta}} (\mu_1 h^i + \mu_2 \bar{h}^i)$, where

$$B_i = \left(\frac{-2}{u^2} \right) \beta_{s_{i0}} + (2) \eta_{s_{i0}} + \frac{1}{\rho} \left(\left(\frac{2}{u^3} \right) \beta_{r_{00}} \right) h_i,$$

$$B = \left(\frac{-1}{u^2} \right) \beta_{r_{00}} + \eta_{r_{00}},$$

$$B_i b^i = \left(\frac{-2}{u^2} \right) \beta_{s_0} + (2) \eta_{(\bar{s}_0)} + \frac{1}{\rho} \left[\left(\frac{2}{u^3} \right) \beta_{r_{00}} \right] (b^2 - u^2),$$

$$B_i \eta^i = \left(\frac{-2}{u^2}\right) \beta_{\bar{s}_0} + (2) \eta_{s_0} + \frac{1}{\rho} \left[\left(\frac{2}{u^3}\right) \beta_{r_{00}}\right] (\theta - u\bar{u}).$$

Let $D^i = 2\bar{G}^i - 2G^i$, from which we obtain the following.

Proposition 9.4.2. *The spray coefficients G^i are connected to ${}^\rho G^i$*

$$G^i = {}^\rho G^i + \frac{\rho}{A} \left[\left(\frac{-2}{u^3 \bar{u}}\right) \beta_{s_0^i} - \left(\frac{1}{u^2 \bar{u}^2}\right) \eta_{s_0^i} \right] + \frac{1}{2\Gamma} [\Gamma_1 b^i + \Gamma_2 \eta^i + \frac{1}{\Theta} \Gamma_3 \rho^i], \quad (9.4.18)$$

where,

$$\begin{cases} \Gamma = \frac{2}{u} + (b^2 - u^2) \frac{2}{u^3}, \\ \Gamma_1 = \left[\frac{2}{u^3}\right] R^\beta, \\ \Gamma_2 = 0, \\ \Gamma_3 = \left\{ \frac{-4}{u^3} - \frac{2\bar{u}}{u^2} + \left(\frac{-2}{u^3}\right) (\theta - u\bar{u}) \right\} R^\beta + \left\{ \frac{2}{u} + \frac{2}{u^3} (b^2 - u^2) \right\} R^\eta. \end{cases} \quad (9.4.19)$$

and

$$\begin{aligned} R^\beta &:= \beta_{r_{00}} - (\rho u) \left\{ \left(\frac{-1}{u^2}\right) \beta_{s_0} + \eta_{\bar{s}_0} \right\}; \\ R^\eta &:= \eta_{r_{00}} - (\rho u) \left\{ \left(\frac{-1}{u^2}\right) \beta_{\bar{s}_0} + \eta_{s_0} \right\}. \end{aligned}$$

□

9.5 Projectively Flat (ρ, β, η) -Metrics

Example 9.5.1. We assume a (ρ, β, η) -metric in the following form: $F = \frac{\rho^2}{\beta} + \eta$, $\Theta(u, \bar{u}) = \frac{1}{u} + \bar{u}$. Let $c_1 > 0$ and $l_1 > 0$ be the largest numbers such that

$$\Theta = \frac{2}{u}, \quad \Gamma = \frac{2b^2}{u^3} > 0. \quad |u| < b < c_1, \quad |\bar{u}| < g < l_1. \quad (9.5.1)$$

Lemma 9.5.2. A (ρ, β, η) -metric $F = \rho \Theta(u, \bar{u})$, where $\Theta(u, \bar{u}) = \frac{1}{u} + \bar{u}$, $u = \frac{\beta}{\rho}$ and $\bar{u} = \frac{\eta}{\rho}$,

is projectively flat on an open subset $u \subseteq R^n$ if and only if

$${}^{\rho}h_{ij}{}^{\rho}G^i + \frac{\beta}{2} \left[\frac{-\rho^2}{\beta^2} \beta_{s_{j0}} + \eta_{s_{j0}} \right] + \frac{2\rho^6}{4b^2\beta^7} [\beta\beta_{r_{00}} + \rho^2\beta_{s_0} - \beta^2\eta_{\bar{u}}] h_j = 0. \quad (9.5.2)$$

Where η_1 and η_2 are given by (9.4.19) and ${}^{\rho}h_{ij} = a_{ij} - \rho_i\rho_j$.

Proof. Let $F = \rho\Theta(u, \bar{u})$ be a projectively flat metric on U . Then, we have

$$G^i = Qy^i \quad (9.5.3)$$

Contracting (9.5.3) with ${}^{\rho}h_{ij}$ and using (9.4.18), we obtain (9.5.2).

On the other hand, let us assume that (9.5.2) holds. Applying the contraction of (9.5.2) with a^{ij} results in

$$\frac{\beta}{2} \left[\frac{-\rho^2}{\beta^2} \beta_{s_{j0}} + \eta_{s_{j0}} \right] = \frac{-2\rho^6}{4b^2\beta^7} [\beta\beta_{r_{00}} + \rho^2\beta_{s_0} - \beta^2\eta_{\bar{u}}] h_j - [{}^{\rho}G^i - \rho G^r \rho_r \rho^i].$$

Substituting this into (9.4.18) results in

$$G^i = \left\{ {}^{\rho}G^r \rho_r + \frac{1}{2\Gamma} [u\Gamma_1 + \bar{u}\Gamma_2 + \frac{1}{\Theta}\Gamma_3] \right\} \rho^i.$$

This suggests that F is projectively flat.

Lemma 9.5.3. *The (ρ, β, η) -metric $F = \frac{\rho^2}{\beta} + \eta$ is locally projectively flat if and only if β is parallel with respect to ρ and η is closed.*

Furthermore, β is parallel with respect to ρ if and only if $b_{i|j} = 0$, which implies both $\beta_{s_{ij}} = 0$ and $\beta_{r_{ij}} = 0$.

Proof. Let $F = \frac{\rho^2}{\beta} + \eta$ be a locally projectively flat metric. Substituting (9.5.1) into (9.5.2)

$${}^{\rho}h_{ij}{}^{\rho}G^i + \frac{1}{2\beta} [-\rho^2\beta_{s_{j0}} + \beta^2\eta_{s_{j0}}] + \frac{2\rho^6}{4b^2\beta^7} [\beta\beta_{r_{00}} + \beta_{s_0}\rho^2 - \beta^2\eta_{\bar{s}_0}] h_j = 0.$$

By multiplying this equation by $8b^2\beta^8\rho^2$, we obtain

$$\begin{cases} 8b^2\beta^8(a_{ij}\rho^2 - y_i y_j)\rho G^i + 4b^2\beta^7\rho^2[-\rho^2\beta_{s_{j0}} + \beta^2\eta_{s_{j0}}] + 4\rho^8\beta[\beta\beta_{r_{00}} + \beta_{s_0}\rho^2 - \\ \beta^2\eta_{\bar{s}_0}]h_j = 0. \end{cases}$$

From above equation we take term which does not contain ρ^2 is;

$$-8b^2\beta^8 y_i y_j \rho G^i = 0.$$

So we know that $-8b^2\beta^8 y_i y_j \neq 0$ so we have

$$\rho G^i = 0.$$

Put value in above equation and we get;

$$4b^2\beta^7[-\rho^2\beta_{s_{j0}} + \beta^2\eta_{s_{j0}}] + 4\rho^6\beta[\beta\beta_{r_{00}} + \beta_{s_0}\rho^2 - \beta^2\eta_{\bar{s}_0}]h_j = 0.$$

From above equation term which does not contain ρ^2 is

$$4b^2\beta^9\eta_{s_{j0}} = 0.$$

So we have,

$$\eta_{s_{j0}} = 0.$$

Put value in above equation and we get;

$$-4b^2\beta^7\beta_{s_{j0}} + 4\rho^4\beta[\beta\beta_{r_{00}} + \beta_{s_0}\rho^2 - \beta^2\eta_{\bar{s}_0}]h_j = 0.$$

From above equation we have term which does not contain ρ^2 is

$$-4b^2\beta^7\beta_{s_{j0}} = 0$$

so we have

$$\beta_{s_{j0}} = 0.$$

Thus we get $\beta_{s_{ij}} = 0$, $\eta_{s_{ij}} = 0$. □

□

9.6 Douglas Spaces by (ρ, β, η) -Metrics

Douglas presented a collection of local functions D_{jkl}^i on TM_0 , which are characterized by

$$D_{jkl}^i := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left(G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i \right).$$

It is straightforward to check that the object

$$D := D_{jkl}^i dx^j \otimes \frac{\partial}{\partial x^i} \otimes dx^k \otimes dx^l$$

The Douglas tensor is a well-defined tensor on TM_0 . A Finsler manifold (M, F) is termed a *Douglas space* if and only if the expression $G^i y^j - G^j y^i$ constitutes a homogeneous polynomial of degree three in y^i [14].

Furthermore, from equation (9.4.18), one can decompose this difference as

$$G^i y^j - G^j y^i = (\rho G^i y^j - \rho G^j y^i) + B^{ij},$$

where

$$\begin{aligned} B^{ij} := & \frac{\rho}{\Theta} [\Theta_u (\beta_{s_0^i} y^j - \beta_{s_0^j} y^i) + \Theta_{\bar{u}} (\eta_{s_0^i} y^j - \eta_{s_0^j} y^i)] + \frac{1}{2\Gamma} \{ [\Theta_{uu} + (g^2 - \bar{u}^2)J] R^\beta \\ & + [\Theta_{u\bar{u}} - (\theta - u\bar{u})J] R^\eta \} (b^i y^j - b^j y^i) + \frac{1}{2\Gamma} \{ [\Theta_{\bar{u}\bar{u}} + (b^2 - u^2)J] R^\eta \\ & + [\Theta_{u\bar{u}} - (\theta - u\bar{u})J] R^\beta \} (\eta^i y^j - \eta^j y^i). \end{aligned} \quad (9.6.1)$$

Proposition 9.6.1. Consider F , the metric introduced in Example (9.5.1). We can prove (ρ, β, η) -metric $F = \frac{\rho^2}{\beta} + \eta$ is Douglas if and only if β is parallel with respect to ρ and η is closed.

Proof. Substituting (9.5.1) into (9.6.1) implies that

$$B^{ij} := \frac{1}{16b^2\beta^8} \left(-8\rho^2 b^2 \beta^7 (\beta_{u_0^i} y^j - \beta_{u_0^j} y^i) + 8b^2 \beta^9 (\eta_{u_0^i} y^j - \eta_{u_0^j} y^i) + 8\rho^6 \beta \right. \quad (9.6.2) \\ \left. (\beta \beta_{r_{00}} + \rho^2 \beta_{u_0} - \beta^2 \eta_{\bar{u}_0}) (b^i y^j - b^j y^i) \right)$$

So we have,

$$16b^2 \beta^8 B^{ij} = -8\rho^2 b^2 \beta^7 (\beta_{u_0^i} y^j - \beta_{u_0^j} y^i) + 8b^2 \beta^9 (\eta_{u_0^i} y^j - \eta_{u_0^j} y^i) + 8\rho^6 \beta \quad (9.6.3) \\ (\beta \beta_{r_{00}} + \rho^2 \beta_{u_0} - \beta^2 \eta_{\bar{u}_0}) (b^i y^j - b^j y^i).$$

Now contracting above equation by y_j

So we have;

$$0 = -8\rho^2 b^2 \beta^7 (\beta_{s_0^i} y^j y_j - \beta_{u_0^j} y^i y_j) + 8b^2 \beta^9 (\eta_{u_0^i} y^j y_j - \eta_{u_0^j} y^i y_j) + 8\rho^6 \beta \quad (9.6.4) \\ (\beta \beta_{r_{00} y_j} + \rho^2 \beta_{u_0 y_j} - \beta^2 \eta_{\bar{u}_0 y_j}) (b^i y^j - b^j y^i).$$

so from above equation we have

$$0 = -8\rho^2 b^2 \beta^7 (a^{ij} \beta_{u_{ij} 0} - \rho^2 \beta_{u_{ij}} y_j) + 8b^2 \beta^9 (a^{ij} \eta_{s_{j0}} - \rho^2 \eta_{u_{ij}} y_j) + 8\rho^6 \beta \quad (9.6.5) \\ (\beta \beta_{r_{ij} y^i} + \rho^2 \beta_{u_j} - \beta^2 y^i \eta_{s_{i0}} b^i) (b^i y^j - b^j y^i).$$

So term which does not contain ρ^2 is;

$$8b^2\beta^9a^{ij}\eta_{sj0} = 0. \quad (9.6.6)$$

This implies

$$\eta_{sj0} = 0. \implies \eta_{sij} = 0.$$

Put all value in above equation and we get

$$0 = -8b^2\beta^7(a^{ij}\beta_{sj0} - \rho^2\beta_{sij}y_j) + 8\rho^4\beta(\beta\beta_{rij}y^i + \rho^2\beta_{sj} - \beta^2y^i\eta_{si0}b^i)(b^iy^j - b^jy^i) \quad (9.6.7)$$

So term which does not contain ρ^2 is;

$$-8b^2\beta^7a^{ij}\beta_{sj0} = 0. \implies \beta_{sj0} = 0.$$

Hence we have;

$$\eta_{sj0} = \beta_{sj0} = 0. \implies \beta_{sij} = \eta_{sij} = 0.$$

Hence theorem. □

CHAPTER 10

Conclusion and Future Scope

10.1 Conclusion

The final chapter consolidates the extensive investigation into the differential geometry of Finsler spaces equipped with special metrics, emphasizing the substantial progress achieved throughout this thesis. The introduction of novel metric classes, including exponential (α, β) -metric, cubic (α, β) -metric, generalized Matsumoto metric, and multi-form based structures, has significantly broadened the traditional framework of (α, β) -metrics, thereby providing a more comprehensive foundation for examining geometric and curvature properties.

Projective changes play a significant role in preserving the intrinsic geometric structure while allowing certain properties such as size, scale, and curvature to vary. Analyzing the behavior of Finsler spaces under these transformations provides deeper insight into their geometric and topological characteristics. In this study, we established the projective transformation between the cubic (α, β) -metric $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ and the Kropina metric $\tilde{F} = \frac{\alpha^2}{\beta}$. Additionally, we examined the projective transformation from the square root (α, β) -metric $F = \sqrt{\alpha(\alpha+\beta)}$ to the Kropina metric $\tilde{F} = \frac{\alpha^2}{\beta}$. Furthermore, the S-curvature properties associated with both of these metrics were analyzed and discussed in detail. We analyzed an exponential (α, β) -metric, $F = \alpha e^{\beta/\alpha} + \beta e^{-\beta/\alpha}$, to determine when it satisfies the criteria for Berwald, Douglas, and Weakly-Berwald spaces. By establishing these specific geometric conditions, this research clarifies how such exponential (α, β) -metric fit within the broader classification of Finsler geometry.

One of the most essential challenges in Finsler geometry is to examine and characterize

Finsler metrics with constant flag curvature. We focused on the classification of locally projectively flat cubic (α, β) -metric and generalized Matsumoto-metric with constant Ricci curvature. In particular, this research seeks to establish the existence and properties of such metrics when the flag curvature is identically zero. We established the conditions under which the cubic (α, β) -metric and the exponential (α, β) -metric are locally dually flat and projectively flat. Furthermore, we characterized these conditions through the use of isotropic S -curvature. Moreover, we derived S -curvature, mean Berwald curvature for exponential (α, β) -metric.

In the present work, we introduced a novel class of Finsler metrics that extend beyond the traditional (α, β) -metrics framework. By considering the Randers metric $F = \alpha + \gamma$, where α is a Riemannian metric and $\gamma = d_i(x)y^i$ is a 1-form on the manifold M , we defined a new metric of the form

$$\tilde{F} = \frac{F^2}{F + \beta} = F\phi(s),$$

with $\phi(s) = \frac{1}{1+s}$ and $s = \frac{\beta}{F}$. The derivation of the spray coefficients and the Cartan connection for these metrics was carried out, providing a comprehensive geometric characterization. Moreover, we established a necessary and sufficient condition for these metrics to be projective. Under suitable conditions, this framework facilitates the generation of a significant family of new Douglas metrics derived from the original metric, thereby enriching the class of Finsler structures available for pattern evaluation and further geometric analysis.

Within the framework of this work, we analyzed $\mathbb{R}$ -complex Finsler spaces equipped with a cubic (α, β) -metric, resulting in the complete derivation of its fundamental metric fields. We derived the fundamental metric fields $\tilde{g}_{ij}$ and $\tilde{g}_{i\bar{j}}$, along with their determinants and inverse of these fundamental metric tensor fields for the Finsler space with cubic (α, β) -metric. Additionally, we established the conditions under which the $\mathbb{R}$ -complex Finsler space with cubic (α, β) -metric exhibits characteristics of a non-Hermitian Finsler space.

Additionally, we introduced a novel class of Finsler metrics by extending (ρ, β) -metrics to incorporate two 1-forms, β and η , on a Riemannian manifold. Specifically, we studied the Kropina–Randers metric $F = \frac{\rho^2}{\beta} + \eta$ and analyzing its fundamental geometric properties. We established key results on the metric structure, derived the spray coefficients, examined conditions for locally projective flatness, and characterized the metric as a Douglas space. Future work may focus on generalizing various (ρ, β) -metrics to (ρ, β, η) -metric, enabling a deeper exploration of curvature properties, projective geometry, and the development of richer structures within Finsler spaces.

10.2 Future Scope

The study of Finsler geometry represents a vibrant and evolving branch of modern mathematics. Current research primarily concentrates on Finsler metrics, with a particular emphasis on the (α, β) -metric a structure defined by a Riemannian metric α and a differential 1-form β . Our primary research interest lies in Finsler metrics, with a specific focus on the (α, β) -metric. While numerous scholars have already leveraged this metric to achieve significant breakthroughs in Finsler geometry, there is still considerable potential for expansion. By applying the (α, β) -metric to diverse and emerging notions within Finsler spaces, future research can delve into the following areas:

- Future research can extend the study of projective transformations to broader classes of (α, β) -metrics, including exponential (α, β) -metric, cubic (α, β) -metric, and generalized Matsumoto metrics, to develop unified geometric frameworks and uncover new curvature and topological properties.
- Investigations into the classification and characterization of Finsler spaces with constant flag curvature can be deepened by exploring additional metric types and their projective flatness, isotropic S -curvature conditions, and dual flatness properties.
- The study of $\mathbb{R}$ -complex Finsler spaces with cubic (α, β) -metric can be expanded by

examining their Hermitian and non-Hermitian properties in higher dimensions and by exploring their implications in complex differential geometry and related fields.

- Generalizing (ρ, β) -metrics to (ρ, β, η) -metrics presents opportunities to investigate the interplay of multiple 1-forms on curvature, projective geometry, and metric structures, potentially leading to richer classes of Finsler spaces with novel geometric and physical interpretations.

- Applying these advanced Finsler metric frameworks to problems in theoretical physics and cosmology, such as gravitational wave modeling, anisotropic space time structures, and generalized gravitational theories can bridge abstract geometry with observable phenomena.

References

- [1] Abbas, S., & Kozma, L. (2023). On generalized Douglas curvature of Finsler metrics. *Global Journal of Advanced Research on Classical and Modern Geometries*, 12, 229–236.
- [2] Aikou, T. (2004). On Finsler geometry on complex vector bundles. *Riemann-Finsler Geometry*, 50, 83–105.
- [3] Aikou, T. (2003). Projective flatness of complex Finsler metrics. *Publicationes Mathematicae Debrecen*, 63, 343–362.
- [4] Akabar Zadeh, H. (1963). Les espaces de Finsler et certaines de leurs généralisations. *Annales Scientifiques de l'École Normale Supérieure*, 3, 1–79.
- [5] Aldea, N., & Campean, G. (2015). On some classes of R -complex Hermitian Finsler spaces. *Journal of the Korean Mathematical Society*, 52(3), 587–601.
- [6] Aldea, N., & Munteanu, G. (2007). (α, β) -complex Finsler metrics. In *Proceedings of the 4th International Colloquium Mathematics in Engineering and Numerical Physics* (Vol. 14, pp. 1–6). BSG Proceedings, Geometry Balkan Press.
- [7] Aldea, N., & Munteanu, G. (2009). On complex Finsler spaces with Randers metric. *Journal of the Korean Mathematical Society*, 46(5), 949–966.
- [8] Amari, S. (1985). *Differential-geometrical methods in statistics* (Lecture Notes in Statistics, Vol. 28). Springer-Verlag.
- [9] Amari, S., & Nagaoka, H. (2000). *Methods of information geometry* (AMS Translations of Mathematical Monographs, Vol. 191). Oxford University Press.
- [10] Antonelli, P. L. (Ed.). (2003). *Handbook of Finsler geometry*. Kluwer Academic Publishers.

- [11] Antonelli, P. L., Ingarden, R. S., & Matsumoto, M. (1993). *The theory of sprays and Finsler spaces with applications in physics and biology*. Kluwer Academic Publishers.
- [12] Bacso, S., & Szilagi, B. (2002). On a weakly-Berwald space of Kropina type. *Mathematica Pannonica*, 13, 91–95.
- [13] Barati, F. (2023). On class of square Finsler metrics. *Journal of Finsler Geometry and its Applications*, 4(2), 74–91.
- [14] Báscó, S., Cheng, X., & Shen, Z. (2007). Curvature properties of (α, β) -metrics. In S. Sabau & H. Shimada (Eds.), *Finsler geometry, Sapporo 2005 - In memory of Makoto Matsumoto* (Advanced Studies in Pure Mathematics, Vol. 48, pp. 73–110). Mathematical Society of Japan.
- [15] Báscó, S., & Matsumoto, M. (1994). Projective change between Finsler spaces with (α, β) -metric. *Tensor N.S.*, 55, 252–257.
- [16] Barthel, W. (1953). Zum Inhaltsbegriff in der Minkowskischen Geometrie. *Mathematische Zeitschrift*, 58, 358–375.
- [17] Báscó, S., & Matsumoto, M. (1997). On the Finsler spaces of Douglas type: A generalization of the notion of Berwald space. *Publ. Math. Debrecen*, 51, 385–406.
- [18] Bao, D., Chern, S. S., & Shen, Z. (2000). *An introduction to Riemann-Finsler geometry*. Springer-Verlag.
- [19] Berwald, L. (1928). Parallelübertragung in allgemeinen Räumen. *Atti Congr. Intern. Mat. Bologna*, 4, 263–270.
- [20] Berwald, L. (1929). Über eine charakteristische Eigenschaft der allgemeinen Räume konstanter Krümmung mit geradlinigen Extremalen. *Monatshefte für Mathematik und Physik*, 36, 315–330.

- [21] Berwald, L. (1926). Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus. *Mathematische Zeitschrift*, 25, 40–73.
- [22] Berwald, L. (1929). Über die n -dimensionalen Geometrien konstanter Krümmung, in denen die Geraden die kürzesten sind. *Mathematische Zeitschrift*, 30, 449–469.
- [23] Berwald, L. (1947). Über Finslersche und Cartansche Geometrie IV, projektive Krümmung allgemeiner affiner Räume und Finslerscher Räume skalarer Krümmung. *Annals of Mathematics*, 48, 755–781.
- [24] Bryant, R. (1996). Finsler structures on the 2-sphere satisfying $K = 1$. In *Finsler geometry* (Contemporary Mathematics, Vol. 196, pp. 27–42). American Mathematical Society.
- [25] Bryant, R. (1997). Projectively flat Finsler 2-spheres of constant curvature. *Selecta Mathematica, New Series*, 3, 161–204.
- [26] Cartan, E. (1914). *Les espaces de Finsler* (Vol. 79). Hermann.
- [27] Chandru, K., & Narasimhamurthy, S. K. (2017). On curvatures of homogeneous Finsler-Kropina spaces. *Gulf Journal of Mathematics*, 5(1), 73–83.
- [28] Ceyhan, S., & Civi, G. (2016). On Z-projective change of Kropina spaces. *International Electronic Journal of Geometry*, 9, 30–37.
- [29] Chen, G., He, Q., & Pan, S. (2014). On weak Berwald (α, β) -metrics of scalar flag curvature, *Journal of Geometry and Physics*, 86, 112–121.
- [30] Chen, X., Mo, X., & Shen, Z. (2003). On the flag curvature of Finsler metrics of scalar curvature. *Journal of the London Mathematical Society*, 68, 762–780.
- [31] Chen, G., & Liu, L. (2016). On dually flat Kropina metrics. *International Journal of Mathematics*, 27(12), 1–13.

- [32] Chen, G., Liu, L., & Jiang, Q. (2019). An important class of conformally flat weak Einstein cubic (α, β) -metrics. *International Journal of Geometric Methods in Modern Physics*, 16(7), 1–10.
- [33] Chen, G., & Liu, L. (2017). On Randers changes of m -th root Finsler metrics $L = \sqrt[m]{A}$ without irreducibility of A . *Annales Polonici Mathematici*, 119, 239–253.
- [34] Cheng, X., & Shen, Z. (2012). *Finsler geometry: An approach via Randers spaces*. Springer.
- [35] Cheng, X., & Tian, Y. (2011). Locally dually flat Finsler metrics with special curvature properties. *Differential Geometry and its Applications*, 29, S98–S106.
- [36] Cheng, X., Shen, Z., & Zhou, Y. (2005). On a class of locally dually flat Finsler metrics. *Journal of Mathematics*, 21(11), 1–13.
- [37] Cheng, X., Shen, Z., & Zhou, Y. (2010). On locally dually flat Randers metrics. *International Journal of Mathematics*, 21, 1531–1543.
- [38] Cheng, X., & Shen, Z. (2009). A class of Finsler metrics with isotropic S -curvature. *Israel Journal of Mathematics*, 169, 317–340.
- [39] Chern, S. S., & Shen, Z. (2005). *Riemann-Finsler geometry* (Nankai Tracts in Mathematics, Vol. 6). World Scientific Publishing.
- [40] Chern, S. S. (1996). Finsler geometry is just Riemannian geometry without quadratic restriction. *Notices of the AMS*, 43(9), 959–963.
- [41] Cui, N. (2006). On the s -curvature of some (α, β) -metrics. *Acta Mathematica Scientia*, 26A(7), 1047–1056.
- [42] Cui, N., & Shen, Y.-B. (2009). Projective change between two classes of (α, β) -metrics. *Differential Geometry and its Applications*, 27, 566–573.
- [43] Delcke, A. (1953). Über die Finsler-Räume mit $A_i = 0$. *Archiv der Mathematik*, 4, 45–51.

- [44] Deng, S. (2009). The S -curvature of homogeneous Randers spaces. *Differential Geometry and its Applications*, 27, 75–84.
- [45] Deng, S., & Wang, X. (2010). The S -curvature of homogeneous (α, β) -metrics. *Balkan Journal of Geometry and Its Applications*, 15(2), 39–48.
- [46] Deng, S. (2012). *Homogeneous Finsler spaces*. Springer.
- [47] Douglas, J. (1928). The general geometry of paths. *Annals of Mathematics*, 29, 143–168.
- [48] Ehresmann, C. (1943). On bundled spaces associated with a differentiable manifold, *Comptes Rendus de l'Académie des Sciences de Paris*, 216, 628–630.
- [49] Finsler, P. (1951). *Über Kurven und Flächen in allgemeinen Räumen* (Doctoral dissertation, Göttingen, 1918). Birkhäuser Verlag.
- [50] Funk, P. (1929). Über Geometrien, bei denen die Geraden die kürzesten sind. *Mathematische Annalen*, 101, 226–237.
- [51] Funk, P. (1936). Über zweidimensionale Finslersche Räume, insbesondere über solche mit geradlinigen Extremalen und positiver konstanter Krümmung. *Mathematische Zeitschrift*, 40, 86–93.
- [52] Gardner, R., & Wilkens, G. (1996). A pseudo-group isomorphism between control systems and certain generalized Finsler structures. *Contemporary Mathematics*, 196, 231–244.
- [53] Gupta, M. K., & Pandey, P. N. (2015). Finsler space subjected to a Kropina change with an h -vector. *Facta Universitatis, Series: Mathematics and Informatics*, 30(4), 513–525.
- [54] Hama, R., Harko, T., & Sabau, S. V. (2023). Conformal gravitational theories in Barthel–Kropina-type Finslerian geometry, and their cosmological implications. *European Physical Journal C*, 83, 1030.

- [55] Hamel, G. (1903). Über die Geometrien, in denen die Geraden die Kürzesten sind. *Mathematische Annalen*, 57, 231–264.
- [56] Hashiguchi, M. (1976). On conformal transformations of Finsler metrics. *Journal of Mathematics of Kyoto University*, 16(1), 25–50.
- [57] Hashiguchi, M., Hojo, S., & Matsumoto, M. (1973). On Landsberg spaces of two dimensions with (α, β) -metric. *Journal of the Korean Mathematical Society*, 10, 17–26.
- [58] Hashiguchi, M., & Ichijyo, Y. (1980). Randers space with rectilinear geodesics. *Reports of the Faculty of Science of Kagoshima University (Mathematics, Physics, and Chemistry)*, 13, 33–40.
- [59] Hu, Z., & Deng, S. (2012). Homogeneous Randers spaces with isotropic S -curvature and positive flag curvature. *Mathematische Zeitschrift*, 270, 989–1009.
- [60] Izumi, H. (1977). Conformal transformations of Finsler spaces. *Tensor, N.S.*, 31, 33–41.
- [61] Khan S., (2025). Some investigation on conformal and projective change in Finsler spaces and its applications. PhD thesis. Gujarat Technological University.
- [62] Khan, S., & Tripathi, B. K. (2025). On conformal vector fields of Finsler-Matsumoto type metric. *Honam Mathematical Journal*, 47(2), 247–257.
- [63] Kim, B. D., & Choi, E. S. (1999). Homogeneous function and its application in a Finsler space, *Communications of the Korean Mathematical Society*, 14(2), 385–392.
- [64] Kosambi, D. D. (1935). Homogeneous metrics. *Proceedings of the Indian Academy of Sciences (Mathematical Sciences)*, 1, 952–954.
- [65] Kobayashi, S. (1975). Negative vector bundles and complex Finsler structures. *Nagoya Mathematical Journal*, 57, 153–166.

- [66] Kropina, V. K. (1959). On projective Finsler spaces with a metric of some special form. *Nauchn. Dokl. Vyass. Skoly, Fiz.-Mat. Nauki*, 2, 38–42.
- [67] Kron, G. (1934). Non-Riemannian dynamics of rotating electrical machinery. *Studies in Applied Mathematics*, 13, 103–194.
- [68] Kropina, V. K. (1961). On projective two-dimensional Finsler spaces with a special metric. *Trudy Sem. Vektor. Tenzor. Anal.*, 11, 277–292.
- [69] Kumar, P., & Chaubey, V. K. (2025). Geometric characteristics of the hypersurface in a Finsler space with Randers β -transformed homogeneous metric. *Honam Mathematical Journal*, 47(4), 540–552.
- [70] Kumar, P., & Tripathi, B. K. (2019). Finsler spaces with some special (α, β) -metric of Douglas type. *Malaya Journal of Matematik*, 7, 132–137.
- [71] Kumar, P., S., M. T., & Ramesha, M. (2016). Projective equivalence between two families of Finsler metrics. *Gulf Journal of Mathematics*, 4(1), 65–74.
- [72] Laugwitz, D. (1955). Zur geometrischen Begründung der Parallelverschiebung in Finslerschen Räumen. *Archiv der Mathematik*, 6, 448–453.
- [73] Lee, Y. (2008). Douglas space of the second kind of Finsler space with a Matsumoto metric. *Journal of the Chungcheong Mathematical Society*, 21, 210–221.
- [74] Lee, I. Y., & Park, H. S. (2004). Finsler spaces with infinite series (α, β) -metric. *Journal of the Korean Mathematical Society*, 41(3), 567–589.
- [75] Lee, I. Y., & Lee, M. H. (2006). On weakly-Berwald spaces of special (α, β) -metric. *Bulletin of the Korean Mathematical Society*, 43(2), 425–441.
- [76] Lin, Z. (2009). Some results on a class of (α, β) -metrics with constant flag curvature. *Differential Geometry and its Applications*, 27(2), 185–198.
- [77] Li, B., & Shen, Z. (2015). On a class of projectively flat Finsler metrics. *arXiv preprint*, arXiv:1510.06303.

- [78] Ma, Y., Zhang, X., & Zhang, M. (2024). Kropina metrics with isotropic scalar curvature via navigation data. *Mathematics*, 12(4), 505.
- [79] Matsumoto, M. (1991). The Berwald connection of a Finsler space with an (α, β) -metric. *Tensor (N.S.)*, 50(1), 18–21.
- [80] Matsumoto, M. (1980). Finsler spaces with (α, β) -metric of Douglas type. *Tensor N.S.*, 60(2), 123–134.
- [81] Matsumoto, M., & Wei, X. (1994). Projective changes of Finsler spaces of constant curvature. *Publ. Math. Debrecen*, 44, 175–181.
- [82] Matsumoto, M. (1992). Theory of Finsler spaces with (α, β) -metric. *Reports on Mathematical Physics*, 31, 43–83.
- [83] Matsumoto, M. (1989). A slope of mountain is a Finsler surface with respect to time measure. *Journal of Mathematics of Kyoto University*, 29(1), 17–25.
- [84] Matsumoto, M. (1974). On Finsler spaces with Randers metric and special forms of important tensors. *Journal of Mathematics of Kyoto University*, 14(3), 477–498.
- [85] Matsumoto, M. (1986). *Foundations of Finsler geometry and special Finsler spaces*. Kaiseisha Press.
- [86] Matsumoto, M. (1972). On a C-reducible Finsler space. *Tensor, N.S.*, 24, 29–37.
- [87] Matsumoto, M. (1980). Projective changes of Finsler metrics and projectively flat Finsler spaces. *Tensor, N.S.*, 34, 305–315.
- [88] Matsumoto, M. (1971). On transformations of locally Minkowskian space. *Tensor, N.S.*, 22, 103–111.
- [89] Munteanu, G. (2004). *Complex spaces in Finsler, Lagrange and Hamilton geometries*. Springer.
- [90] Munteanu, G., & Purcaru, M. (2009). On R -complex Finsler spaces. *Balkan Journal of Geometry and Its Applications*, 14(1), 52–59.

- [91] Myers, S. B., & Steenrod, N. (1939). The group of isometries of a Riemannian manifold. *Annals of Mathematics*, 40, 400–416.
- [92] Narasimhamurthy, S. K., Latha Kumari, G. N., & Bagewadi, C. S. (2011). Projectively flat Kropina metric with $K = 0$. *Applied Mathematical Sciences*, 5, 585–593.
- [93] Narasimhamurthy, S., Kavyashree, A., Ajith, & Mallikarjun, Y. (2013). On a class of weakly Berwald (α, β) -metrics of scalar flag curvature. *International Journal of Mathematics Research*, 5(1), 97–108.
- [94] Narasimhamurthy, S. (2014). A characterization of locally dually flat first approximate Matsumoto metric. *Balkan Journal of Geometry and Its Applications*, 19, 80–87.
- [95] Narasimhamurthy, S. K., & Vasantha, D. M. (2007). The study of locally dually flat Finsler space with special (α, β) -metric. *Journal of the Tensor Society*, 6, 27–33.
- [96] Park, H. S., & Lee, I.-Y. (1999). On projectively flat Finsler spaces with (α, β) -metric. *Communications of the Korean Mathematical Society*, 14(2), 373–383.
- [97] Pandey, V. K. (2012). *A study on differential geometry of some special Finsler spaces* [Doctoral dissertation, V. B. S. Purvanchal University].
- [98] Park, H. S., & Choi, E. S. (1999). Finsler spaces with an approximate Matsumoto metric of Douglas type. *Communications of the Korean Mathematical Society*, 14, 535–544.
- [99] Park, H. S., Park, H. Y., Kim, B. D., & Choi, E. S. (2003). Projectively flat Finsler space with certain (α, β) -metric. *Bulletin of the Korean Mathematical Society*, 40, 649–661.
- [100] Park, H. S., & Lee, Y. (2008). Projective change between a Finsler space with (α, β) -metric and the associated Riemannian metric. *Canadian Journal of Mathematics*, 60, 443–456.

- [101] Park, H. S., & Choi, E. S. (2002). Finsler space with the second approximate Matsumoto metric. *Bulletin of the Korean Mathematical Society*, 39, 153–163.
- [102] Piscoran, L., & Mishra, L. (2016). Projective change for a new class of (α, β) -metrics. *Mathematica Aeterna*, 6, 885–894.
- [103] Prajapati, S., & Tripathi, B. K. (2025). Weakly Berwald Spaces with Generalized Matsumoto Metrics. *Arab. J. Math.*, 1–10.
- [104] Purcaru, M. (2011). On R -complex Finsler spaces with Kropina metrics. *Bulletin of the Transilvania University of Braşov*, 4(53), 79–88.
- [105] Rapscak, A. (1961). Über die bahntreuen Abbildungen metrischer Räume. *Publ. Math. Debrecen*, 8, 285–290.
- [106] Rajabi, T., & Sadeghzadeh, N. (2021). A new class of Finsler metrics. *Matematički Vesnik*, 73(1), 1–13.
- [107] Randers, G. (1941). On an asymmetric metric in the four-space of general relativity. *Physical Review*, 59, 195–199.
- [108] Rafie-Rad, M., & Rezaei, B. (2012). On Einstein Matsumoto metrics. *Nonlinear Analysis: Real World Applications*, 13(2), 882–886.
- [109] Rizza, G. B. (1963). Strutture di Finsler di tipo quasi Hermitiano. *Rivista di Matematica della Università di Parma*, 4, 83–106.
- [110] Rund, H. (1959). *The differential geometry of Finsler spaces*. Springer-Verlag, Berlin.
- [111] Rund, H. (1951). Über die Parallelverschiebung in Finslerschen Räumen. *Mathematische Zeitschrift*, 54, 115–128.
- [112] Sadeghzadeh, N., & Rajabi, T. (2024). On (α, β, γ) -metrics. *Matematički Vesnik*, 76(3), 159–173.

- [113] Sadeghzadeh, N. (2025). A new class of Finsler metrics: Douglas curvature and its generalizations. *Filomat*, 39(10), 3381–3399.
- [114] Sadeghzadeh, N. (2025). New classes of Finsler metrics: The birth of new projective invariant. *Differential Geometry and its Applications*, 99, 1–18.
- [115] Sadeghzadeh, N. (2025). On a class of Finsler metrics with special Douglas curvature. *Journal of Finsler Geometry and its Applications*, 6(1), 45–58.
- [116] Sadeghzadeh, N., & Rajabi, T. (2020). On a class of (α, β, γ) -metrics with isotropic S -curvature. *Journal of Finsler Geometry and its Applications*, 1(2), 65–77.
- [117] Senarath, P. (2003). *Differential geometry of projectively related Finsler spaces* [Doctoral dissertation, Massey University].
- [118] Shanker, G., & Kaur, K. (2019). Homogeneous Finsler space with infinite series (α, β) -metric. *Applied Sciences*, 21, 219–235.
- [119] Shanker, G., & Kaur, K. (2019). Naturally reductive homogeneous space with an invariant (α, β) -metric. *Lobachevskii Journal of Mathematics*, 40(2), 210–218.
- [120] Shanker, G., & Rani, S. (2020). On S -curvature of a homogeneous Finsler space with square metric. *International Journal of Geometric Methods in Modern Physics*, 17(2), 2050019.
- [121] Shukla, H. S., & Tiwari, A. P. (2012). Conformal correspondence of Finsler spaces with special (α, β) -metric, *Journal of Tensor Society*, 6(1), 35–41.
- [122] Shen, Z. (2001). *Differential geometry of spray and Finsler spaces*. Kluwer Academic Publishers.
- [123] Shibata, C. (1978). On Finsler spaces with Kropina metric. *Reports on Mathematical Physics*, 13(1), 117–128.
- [124] Shen, Z. (2001). On projectively related Einstein metrics in Riemann-Finsler geometry. *Mathematische Annalen*, 320(4), 625–647.

- [125] Shen, Z., & Yildirim, G. C. (2008). On a class of projectively flat metrics with constant flag curvature. *Canadian Journal of Mathematics*, 60, 443–456.
- [126] Shen, Z. (2001). *Lectures on Finsler geometry*. World Scientific.
- [127] Shen, Z. (2003). Projectively flat Randers metric with constant flag curvature. *Mathematische Annalen*, 325, 19–30.
- [128] Shen, Z. (1997). Volume comparison and its applications in Riemann-Finsler geometry. *Advances in Mathematics*, 128, 306–328.
- [129] Shahbazi Nia, M., Tayebi, A., & Peyghan, E. (2015). On Randers changes of m -th root metrics. *International Electronic Journal of Geometry*, 8(1), 14–20.
- [130] Shen, Z. (2006). Riemann-Finsler geometry with applications to information geometry. *Chinese Annals of Mathematics, Series B*, 27(1), 73–94.
- [131] Shibata, C. (1984). On invariant tensors of ϕ -changes of Finsler metrics. *Journal of Mathematics of Kyoto University*, 24(1), 163–188.
- [132] Shen, Z. (2003). Projectively flat Finsler metrics of constant flag curvature. *Transactions of the American Mathematical Society*, 355, 1713–1728.
- [133] Shen, Z. (2003). Finsler metrics with $K = 0$ and $S = 0$. *Canadian Journal of Mathematics*, 55, 112–132.
- [134] Shen, Z. (2005). Non-positively curved Finsler manifolds with constant S -curvature. *Mathematische Zeitschrift*, 249, 625–639.
- [135] Shen, Z. (2013). On some non-Riemannian quantities in Finsler geometry, *Canadian Mathematical Bulletin*, 56, 184–193.
- [136] Shen, Z. (2013). *Differential geometry of spray and Finsler spaces*. Springer.
- [137] Shanker, G., & Baby, S. A. (2016). On R -complex Finsler spaces with special (α, β) -metrics. *International Journal of Pure and Applied Mathematics*, 29(5), 609–619.

- [138] Shanker, G., & Sharma, R. K. (2018). R -complex Finsler spaces with infinite series (α, β) -metric. *International Journal of Pure and Applied Mathematics*, 120(3), 351–363.
- [139] Sulanke, R. (1955). Die eindeutige Bestimmtheit des von Hanno Rund eingeführten Zusammenhangs in Finsler-Räumen. *Wissenschaftliche Zeitschrift der Humboldt-Universität zu Berlin*, 4, 229–233.
- [140] Synge, J. L. (1925). A generalization of the Riemannian line element. *Transactions of the American Mathematical Society*, 27, 61–67.
- [141] Shanker, G., & Kaur, J. (2023). On the Ricci curvature of a homogeneous Finsler space with generalized m -Kropina metric. *Journal of Mathematical Extension*, 17(7), 1–24.
- [142] Tayebi, A., & Shahbazi Nia, M. (2017). Matsumoto change of m -th root Finsler metrics. *Publications de l'Institut Mathématique, Nouvelle Série*, 101(115), 183–190.
- [143] Tayebi, A., Tabatabaeifar, T., & Peyghan, E. (2014). On Kropina change for m -th root Finsler metrics. *Ukrainian Mathematical Journal*, 66(1), 160–164.
- [144] Tayebi, A., & Nia, M. (2015). A new class of projectively flat Finsler metrics with constant flag curvature $K = 1$. *Differential Geometry and its Applications*, 41, 123–133.
- [145] Tayebi, A., & Sadeghi, H. (2014). Some (q, α, β) -metrics projectively related to a Kropina metric. *Global Journal of Advanced Research on Classical and Modern Geometries*, 3(2), 102–114.
- [146] Taylor, J. H. (1925). A generalization of Levi-Civita's parallelism and Frenet formulas. *Transactions of the American Mathematical Society*, 27, 246–264.
- [147] Tiwari, B., & Kumar, M. (2015). On Finsler space with a special (α, β) -metric. *Journal of the Indian Mathematical Society (N.S.)*, 82(3-4), 207–218.

- [148] Tripathi, B. K. (2020). Hypersurfaces of a Finsler space with exponential form of (α, β) -metric. *Annals of the University of Craiova, Mathematics and Computer Science Series*, 47(1), 132–140.
- [149] Tripathi, B. K., & Kumar, P. (2022). Douglas spaces for some (α, β) -metric of a Finsler space. *Journal of Advanced Mathematical Studies*, 15(4), 444–455.
- [150] Tripathi, B. K., Khan, S., & Chaubey, V. K. (2023). On projectively flat Finsler space with a cubic (α, β) -metric. *Filomat*, 37(26), 8975–8982.
- [151] Tripathi, B. K., & Khan, S. (2023). On weakly Berwald space with a special cubic (α, β) -metric. *Surveys in Mathematics and its Applications*, 18, 1–11.
- [152] Tripathi, B. K., & Pandey, K. B. (2016). Characterization of locally dually flat special Finsler (α, β) -metrics. *International Journal of Mathematical Combinatorics*, 4, 44–52.
- [153] Tripathi, B. K., & Khan, S. (2024). On locally projectively flat Finsler space of a special exponential metric with constant flag curvature. *Armenian Journal of Mathematics*, 16(15), 1–11.
- [154] Tiwari, B., & Prajapati, G. K. (2023). On locally dually flat special Finsler (α, β) -metrics. *Acta Mathematica Academiae Paedagogicae Nyíregyháziensis*, 34(1), 21–33.
- [155] Vanithalakshmi, S. M., Narasimhamurthy, S. K., & Roopa, M. K. (2020). On R -complex Finsler space with special (α, β) -metric. *Journal of International Academy of Physical Sciences*, 24, 247–262.
- [156] Vanithalakshmi, S. M., Narasimhamurthy, S. K., & Roopa, M. K. (2018). On R -complex Finsler space with Matsumoto metric. *International Journal of Mathematics and its Applications*, 7(1), 23–30.

- [157] Vanithalakshmi, S. M., & Narasimhamurthy, S. K. (2019). S-curvature of homogeneous Finsler space with special (α, β) -metrics. *Gulf Journal of Mathematics*, 7(1), 45–55.
- [158] Xia, Q. (2011). On locally dually flat (α, β) -metrics. *Differential Geometry and its Applications*, 29(2), 233–243.
- [159] Xia, Q. (2011). On a class of locally dually flat Finsler metrics of scalar flag curvature. *Publicationes Mathematicae Debrecen*, 78, 169–190.
- [160] Xu, M., & Deng, S. (2015). Homogeneous (α, β) -spaces with positive flag curvature and vanishing S -curvature. *Nonlinear Analysis*, 127, 45–54.
- [161] Yang, G. (2018). On a class of Einstein-reversible Finsler metrics. *Differential Geometry and its Applications*, 60, 80–103.
- [162] Yu, C. (2014). On dually flat Randers metrics. *Nonlinear Analysis*, 95, 146–155.
- [163] Yu, C., & Zhu, H. (2011). On a new class of Finsler metrics. *Differential Geometry and its Applications*, 29, 244–254.
- [164] Yu, Y., & You, Y. (2010). On Einstein m -th root metrics. *Differential Geometry and its Applications*, 28, 290–294.
- [165] Zhou, L. (2012). Spherically symmetric Finsler metrics in $\mathbb{R}^n$. *Publicationes Mathematicae Debrecen*, 80(1-2), 67–77.
- [166] Zohrehvand, M., Fasihi-Ramandi, G., & Azami, S. (2024). On Kropina transformation of exponential (α, β) -metrics. *Journal of Finsler Geometry and its Applications*, 5(1), 35–52.
- [167] Zhu, H., & Song, L. (2023). On a class of Finsler metrics with weakly isotropic scalar curvature. *SSRN Electronic Journal*, 1–16.
- [168] Zohrehvand, M. (2024). Projective change between G.R.C exponential metrics and Randers metrics, *Annals of the University Al. I. Cuza of Iasi - Mathematics*, 70(1), 17–29.

List of Publications

1. *On a Weakly-Berwald Space with Exponential Form of (α, β) - Metric*, Journal of Rajasthan Academy of Physical Sciences, **22**, 97-113 (2023). **Web of Science/UGC-Care list**. <https://raops.org.in/epapers/june239.pdf>

2. *Projectively Flat Cubic (α, β) -Metric with Constant Flag Curvature $K=0$* , Surveys in Mathematics and its Applications, **19**, 167-178 (2024). **Scopus/UGC-Care list**. <https://www.utgjiu.ro/math/sma/v19/p1910.pdf>.

3. *Berwald and Douglas Spaces of a Finsler Space With an Exponential form of (α, β) -metric*, Korean Journal of Mathematics, **32** (2024), 661-671. **Web of science/Scopus**. <https://kkms.org/index.php/kjm/article/view/1531>

4. *On $\mathbb{R}$ -Complex Finsler Space with Cubic (α, β) -metric*, Journal of Dynamical Systems and Geometric Theories, **22** (2024), 47-63. **Web of Science**. <https://doi.org/10.47974/JDSGT-2024-003>

5. *Locally dually flatness properties in Cubic (α, β) -metric*, Mathematical Foundations of Computing, **8**(4) (2025) , 507-517. **Scopus, Web of Science**. 10.3934/mfc.2024027

6. *Projective change between Cubic (α, β) -metric and Kropina metric*, Bulletin of Mathematical Analysis and Applications, **17** (1) (2025), 1-16. **Scopus, Web of Science**. <http://www.bmathaa.org/repository/docs/BMAA17-1-1.pdf>.

7. *Projective change between Square root metric and Kropina metric*, Bulletin of the Institute of Mathematics Academia Sinica , **21** (2026), 1-19. **Web of Science**. 10.21915/BIMAS.2026101.

Paper presented in Conferences:

1. Presented a paper entitled *On R-Complex Finsler Space with Matsumoto (α, β) -metric* in the Department of Mathematics North-Eastern Hill University, Shillong, Meghalaya, India, February 15–16, 2024.
2. Presented a paper entitled *On R-Complex Finsler Space with Cubic (α, β) -metric* in the International Geometry Symposium, Munzur University, Turkey, July 1–2, 2024.
3. Presented a paper entitled *Projectively Flat Cubic (α, β) -Metric with Constant Flag Curvature $K = 0$* in the International Conference On Mathematical Modelling, Simulation and Nonlinear Dynamics - 2024, Department of Mathematics, Bharathiar University, Coimbatore, Tamil Nadu, India, October 24–26, 2024.
4. Presented a paper entitled *Projectively Flat Generalized Matsumoto Metric with $K = 0$* in the International Symposium on Differential Geometry and its Application, Munzur University, Turkey, July 3–4, 2025.
5. Presented a paper entitled *Projective Change between Cubic (α, β) -Metric and Kropina Metric* in the International Symposium in Memory of the 100th Anniversary of Gazi University, Turkey, February 2-3, 2026.
6. Presented a paper entitled *On (α, β, γ) -metrics with Kropina Metric* in the International Conference on Mathematics and its Application in Artificial Intelligence and Modeling, Department of Mathematics, Salesian College, Siliguri, India, May 28–30, 2026.